ISRAEL JOURNAL OF MATHEMATICS, Vol. 70, No. 1, 1990

# PRINCIPAL EIGENVALUES, TOPOLOGICAL PRESSURE, AND STOCHASTIC STABILITY OF EQUILIBRIUM STATES

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ABSTRACT
Suppose that $L$ is a second order elliptic differential operator on a manifold $M$, $B$ is a vector field, and $V$ is a continuous function. The paper studies by probabilistic and dynamical systems means the behavior as $\varepsilon \to 0$ of the principal eigenvalue $\lambda^\varepsilon(V)$ for the operator $L_V^\varepsilon = \varepsilon L + (B, \nabla) + V$ considered on a compact manifold or in a bounded domain with zero boundary conditions. Under certain hyperbolicity conditions on invariant sets of the dynamical system generated by the vector field $B$ the limit as $\varepsilon \to 0$ of this principal eigenvalue turns out to be the topological pressure for some function. This gives a natural transition as $\varepsilon \to 0$ from Donsker–Varadhan's variational formula for principal eigenvalues to the variational principle for the topological pressure and unifies previously separate results on random perturbations of dynamical systems.

## 1. Introduction

Let $M$ be a locally compact $\nu$-dimensional $C^2$-class Riemannian manifold and $G \subset M$ be a connected open set in $M$ with a piecewise smooth boundary $\partial G$ such that $\bar{G} = G \cup \partial G$ is compact. We assume that either $\partial G$ consists of more than one point or it is empty, i.e. $G = M$, and in the latter case $M$ is compact. Suppose that $L$ is a second order elliptic differential operator with $C^2$ coefficients which is strongly elliptic in the whole $M$ if $G = M$ or in some neighborhood of $\bar{G}$ in $M$ if $G$ is a proper subset of $M$ and $L1 = 0$. The set up includes also a $C^2$ vector field $B$ on $M$ and a continuous function $V$ on $M$. If $G \neq M$ then the coefficients of $L$ and $B$ and the function $V$ are supposed to be

[†] This work was supported by US–Israel Binational Science Foundation.
Received April 17, 1989

zero outside of some compact set containing $\bar{G}$. This is not a restriction since we will consider boundary value problems in $G$ which depend only on the behavior of coefficients inside $G$. The operator $L^\varepsilon = \varepsilon L + B$ generates diffusion Markov processes $X_t^\varepsilon$ considered as small random perturbations of the flow $F^t : M \to M$ given by

$$\frac{d(F^t x)}{dt}\bigg|_{t=0} = B(x). \tag{1.1}$$

Let $\tau_G^\varepsilon$ be the first exit time of the process $X_t^\varepsilon$ from $G$. Then

$$u_g^\varepsilon(t,x) = T_{V,G}^\varepsilon(t)g(x) \overset{\mathrm{def}}{=} E_x^\varepsilon \chi_{\tau_G^\varepsilon > t}\, g(X_t^\varepsilon)\exp\left(\int_0^t V(X_s^\varepsilon)ds\right) \tag{1.2}$$

solves the problem (see, for instance, [F2], Chapter 6),

$$\begin{cases} \dfrac{\partial u_g^\varepsilon}{\partial t} = L^\varepsilon u_g^\varepsilon + V u_g^\varepsilon, \\ u_g^\varepsilon \big|_{t=0} = g, \quad u_g^\varepsilon \big|_{x\in\partial G} = 0. \end{cases} \tag{1.3}$$

Here $E_x^\varepsilon$ is the expectation for the process $X_t^\varepsilon$ starting at $x$ and $\chi_A = 1$ if $A$ occurs and $= 0$ if not. In the case when $G = M$ one should take $\tau_G^\varepsilon = \infty$ in (1.2) and the boundary condition on $\partial G$ in (1.3) should be removed. The operators $T_{V,G}^\varepsilon(t)$, $t \geqq 0$ form a strongly continuous semigroup of linear operators on the space $C_0(G)$ of continuous functions $g$ on $\bar{G}$ which vanish on the boundary $\partial G$ if $\partial G \neq \varnothing$ with the norms

$$\| g \| = \sup_{x\in G} |g(x)| \quad \text{and} \quad \| T_{V,G}^\varepsilon(t) \| = \sup_{g\in C_0(G),\, \| g \| = 1} \| T_{V,G}^\varepsilon(t) g \| .$$

By the submultiplicative property of the norm, the limit (which, in fact, does not depend on the norm)

$$\lambda_G^\varepsilon(V) = \lim_{t\to\infty} \frac{1}{t} \ln \| T_{V,G}^\varepsilon(t) \| = \inf_{t>0} \frac{1}{t} \ln \| T_{V,G}^\varepsilon(t) \| \tag{1.4}$$

exists and is finite. Then $\lambda_G^\varepsilon(V)$ belongs to the spectrum of $L_V^\varepsilon$ corresponding to zero data on $\partial G$ and no point of this spectrum has bigger real part (see [DV3], Theorem 2.2). If $V$ is Hölder continuous then the spectrum is pure point and in the same way as in Lemma 3.1 of [Ki1] it follows from the theory of positive operators from [Kr] that $\lambda_G^\varepsilon(V)$ is the principal eigenvalue of the operator $L_V^\varepsilon = L^\varepsilon + V$ corresponding to the Dirichlet boundary conditions on $\partial G$ if

$\partial G \neq \varnothing$ which means that all other eigenvalues have smaller real parts. Moreover $\| T^{\varepsilon}_{V,G}(t) \| = \sup_{x\in G} Q^{\varepsilon}_1(t, x, V, G)$, and so

$$(1.5)\qquad \lambda^{\varepsilon}_G(V) = \lim_{t\to\infty} \frac{1}{t} \ln \sup_{x\in G} Q^{\varepsilon}_1(t, x, V, G)$$

where

$$(1.6)\qquad Q^{\varepsilon}_1(t, x, V, G) = E^{\varepsilon}_x \chi_{\tau^{\varepsilon}_G > t} \exp\left(\int_0^t V(X^{\varepsilon}_s)ds\right).$$

If $V$ is Hölder continuous then there exists an eigenfunction $r^{\varepsilon}_{G,V}$ of $L^{\varepsilon}_V$ corresponding to $\lambda^{\varepsilon}_G(V)$ (see [Kr]) which is unique and it is the only positive in $G$ eigenfunction of $L^{\varepsilon}_V$ with zero data on $\partial G$. Then

$$\begin{aligned} Q^{\varepsilon}_1(t, x, V, G) = T^{\varepsilon}_{V,G}(t)1(x) &\geqq \| r^{\varepsilon}_{G,V} \|^{-1} T^{\varepsilon}_{V,G}(t) r^{\varepsilon}_{G,V}(x) \\ &= \| r^{\varepsilon}_{G,V} \|^{-1} e^{t\lambda^{\varepsilon}_G(V)} r^{\varepsilon}_{G,V}(x) \end{aligned}$$

which together with (1.5) implies that for any $x \in G$,

$$(1.7)\qquad \lambda^{\varepsilon}_G(V) = \lim_{t\to\infty} \frac{1}{t} \ln Q^{\varepsilon}_1(t, x, V, G).$$

In view of (1.5) and (1.6)

$$(1.8)\qquad \lambda^{\varepsilon}_G(0) - C_0 \leqq \lambda^{\varepsilon}_G(V) \leqq \lambda^{\varepsilon}_G(0) + C_0, \quad \text{where } C_0 = \sup_{x\in G} |V(x)|.$$

If $G = M$ then always $\lambda^{\varepsilon}_G(0) = 0$ since in this case we assume $M$ to be compact and so $T^{\varepsilon}_{0,M}(t)1 = 1$. On the other hand, if $\bar{G}$ is a proper compact subset of $M$ then I showed in [Ki3] that $\lambda^{\varepsilon}_G(0) \to -\infty$ as $\varepsilon \to 0$ if and only if the dynamical system $F^t$ has no invariant set in $\bar{G}$. By (1.8) the same conclusion holds with respect to $\lambda^{\varepsilon}_G(V)$. Thus the limiting behavior of $\lambda^{\varepsilon}_G(V)$ as $\varepsilon \to 0$ is interesting to study when either $G = M$ and $V \neq 0$ or $\bar{G} \neq M$ and there exist $F^t$-invariant sets in $\bar{G}$. The simplest case when there exist $F^t$-invariant sets in $G$ is, of course, $B \equiv 0$. Then it is easy to see that

$$(1.9)\qquad \lim_{\varepsilon\to 0} \lambda^{\varepsilon}_G(V) = \sup_{x\in G} V(x).$$

Especially nice results one obtains when $\bar{G}$ contains finite number of isolated $F^t$-invariant sets with some kind of hyperbolicity conditions where the topological pressure comes into the picture. Moreover in this case one sees the natural transition as $\varepsilon \to 0$ from Donsker–Varadhan's variational formula for principal eigenvalues to the variational principle for the topological pressure. In [Ki1], [Ki2], and [EK] we treated the case $V \equiv 0$ having in mind small random perturbations of dynamical systems as the main motivation there. In this paper I extend results to any continuous function $V$ which is natural in applications to parabolic equations.

The introduction of a function $V$ in (1.3) enables one to make another step further and generalize results for some quasi-linear equations. By (1.2) and (1.5),

$$\limsup_{t\to\infty} \frac{1}{t} \ln \| u_g^\varepsilon(t,\cdot) \| \leqq \lambda_G^\varepsilon(V) \tag{1.10}$$

for any continuous $g$. On the other hand, if $r_{G,V}^\varepsilon$ is the eigenfunction of $L_V^\varepsilon$ corresponding to $\lambda_G^\varepsilon(V)$ then

$$u_{r_{G,V}^\varepsilon}^\varepsilon(t,x) = e^{t\lambda_G^\varepsilon(V)} r_{G,V}^\varepsilon(x)$$

is the solution of (1.3) with $g = r_{G,V}^\varepsilon$ and we will get an equality in (1.10) for such $g$. Thus $\lambda_G^\varepsilon(V)$ characterizes the maximal growth rate in $t$ of solutions of the problem (1.3). Moreover, for any $g \geqq 0$, $g \not\equiv 0$, $\lim_{t\to\infty}(1/t)\ln u_g^\varepsilon(t,x)$ exists and equals $\lambda_G^\varepsilon(V)$ (see Lemma 4.1). This growth rate can be studied also for quasi-linear equations of the form

$$\begin{cases} \dfrac{\partial w_g^\varepsilon(t,x)}{\partial t} = L^\varepsilon w_g^\varepsilon(t,x) + R(x, w_g^\varepsilon), \\ w_g^\varepsilon \big|_{t=0} = g, \quad w_g^\varepsilon \big|_{x\in\partial G} = 0, \end{cases} \tag{1.11}$$

where

$$V(x,u) = u^{-1} R(x,u), \qquad u \neq 0 \tag{1.12}$$

is supposed to be bounded and continuous, and uniformly in $x \in G$ the limit

$$V_0(x) = V(x,0) = \lim_{u\to\infty} V(x,u) \tag{1.13}$$

exists. In the case $G = M$ the boundary condition in (1.11) should be disregarded. If

$$\sup_{x\in M,u} V(x,u) \leqq C_0 < -\lambda_G^\varepsilon(0) \tag{1.14}$$

for $\varepsilon > 0$ then for such $\varepsilon$ all solutions of (1.11) tend to zero as $t \to \infty$ and we will derive from the linear case that

$$\limsup_{t\to\infty} \frac{1}{t} \ln |w_g^\varepsilon(t, x)| \leqq \lambda_G^\varepsilon(V_0) \tag{1.15}$$

and again for $g \geqq 0$, $g \not\equiv 0$ lim sup can be replaced by lim and the inequality by an equality. Similar result holds true if $\lim_{u\to\infty} V(x, u)$ exists and solutions of (1.11) tend to infinity as $t \to \infty$.

In view of [DV1] and [DV3] one has the following variational representation:

$$\lambda_G^\varepsilon(V) = \sup_{\mu\in\mathscr{P}(\bar G)} \left( \int_{\bar G} V d\mu - I_G^\varepsilon(\mu) \right) \tag{1.16}$$

where $\mathscr{P}(\bar G)$ is the space of probability measures on $G$,

$$I_G^\varepsilon(\mu) = - \inf_{u\in D_+} \int_{\bar G} \frac{L^\varepsilon u}{u} d\mu, \tag{1.17}$$

and $D_+$ is the set of functions from the domain of $L^\varepsilon$ having positive upper and lower bounds. The functional $I_G^\varepsilon(\mu)$ is lower semicontinuous which implies the existence of a measure $\mu_V^\varepsilon$ for which the supremum (1.16) is attained. These generalize invariant measures of the process $X_t^\varepsilon$ which are $\mu_V^\varepsilon$ for $V \equiv 0$ and $M$ compact. It turns out that all limit points as $\varepsilon \to 0$ of measures $\mu_V^\varepsilon$ are invariant with respect to the flow $F^t$. This yields, in particular, that if $G = M$ is compact and $F^t$ is a uniquely ergodic flow on $M$, i.e. there exists just one $F^t$-invariant measure $\mu$, then

$$\lim_{\varepsilon\to 0} \lambda_G^\varepsilon(V) = \int_M V d\mu. \tag{1.18}$$

The topological pressure $P_K(F, \psi)$ of the flow $F^t$ on an $F^t$-invariant set $K$ given a continuous function $\psi$ can be characterized by the variational principle

$$P_K(F, \psi) = \sup_{\mu\in\mathscr{M}_K^F} \left( h_\mu(F^1) + \int_K \psi d\mu \right) \tag{1.19}$$

where $\mathscr{M}_K^F$ is the space of $F^t$-invariant probability measures on $K$ and $h_\mu(F^1)$ denotes the entropy of $F^1$ with respect to $\mu$. We will see that if the limit set of the dynamical system $F^t$ restricted to $\bar G$ consists only of a finite number of hyperbolic sets then $\lambda_G^\varepsilon(V)$ converges as $\varepsilon \to 0$ to $P_K(F, V + \varphi^u)$ where the function $\varphi^u$ is defined by (2.5). This already implies by convexity arguments that all limit points as $\varepsilon \to 0$ of measures $\mu_V^\varepsilon$ are $F^t$-invariant measures on which the supremum in the right hand side of (1.19) with $\psi = V + \varphi^u$ is attained. The measures giving the supremum in (1.19) are called equilibrium states and in the hyperbolic case for each Hölder continuous function $\psi$ there exists just one equilibrium state (see [BR]). So we obtain convergence of $\mu_V^\varepsilon$ to the equilibrium state corresponding to $\varphi^u + V$ if $V$ is Hölder continuous. This generalizes and gives a unified approach to results discussed in Chapter II of [Ki4], as well, as explains why limiting measures are likely to be equilibrium states also in more general circumstances.

The above results can be also obtained for discrete time models where random perturbations may be not necessarily of diffusion type. In the concluding section of this paper, I study corresponding questions for operators given by (1.2) with $\varepsilon = 0$.

I am pleased to thank S. R. S. Varadhan for his valuable remarks concerning Section 3 of this paper. A part of this work was done during my visit to the Institute for Advanced Study at Princeton in February 1989 whose hospitiality and excellent working conditions I gratefully acknowledge.

## 2. Convergence of principal eigenvalues

In this section we will formulate some of the results concerning the behavior of $\lambda_G^\varepsilon(V)$ as $\varepsilon \to 0$ and solutions of (1.3) and (1.11) which will be proved mainly in Section 4.

First, consider the simplest case $B \equiv 0$. Then any subset of $G$ is $F^t$-invariant and so the behavior of $\lambda_G^\varepsilon(V)$ should not be trivial. Let $x_0 \in \bar{G}$ be such that $V(x_0) = \sup_{x \in G} V(x)$. Take the $\varepsilon$-ball $U_\varepsilon(x_0)$ centered at $x_0$. Then by (1.5) and (1.6) applied to $U_\varepsilon(x_0) \cap G$ in place of $G$ we derive $\lambda_G^\varepsilon(V) \geqq \lambda^\varepsilon_{G \cap U_\varepsilon(x_0)}(V)$. Since $V$ is continuous it follows by (1.6),

$$(2.1) \quad Q_1^\varepsilon(t, x, V, U_\varepsilon(x_0) \cap G) \geqq P_x^\varepsilon\{\tau^\varepsilon_{U_\varepsilon(x_0) \cap G} > t\} \exp(t(V(x_0) - \delta(\varepsilon)))$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $P_x^\varepsilon\{\cdot\}$ is the probability of an event in brackets for the process $X_t^\varepsilon$ starting at $x$. Since $B \equiv 0$ the probability in (2.1) decreases not faster than $\exp(-\varepsilon\gamma t)$ for some $\gamma > 0$ provided $x \in U_\varepsilon(x_0) \cap G$ which together with (1.5) gives

$$\liminf_{\varepsilon \to 0} \lambda_G^\varepsilon(V) \geqq V(x_0).$$

On the other hand by (1.6),

$$Q_1^\varepsilon(t, x, V, G) \leqq \exp\left(t \sup_{x \in G} V(x)\right)$$

and we obtain the upper bound which yields (1.9).

In order to simplify notations we assumed in Introduction that $L1 = 0$, i.e. $L$ has the form

$$L = \frac{1}{2} \sum_{i,j}^{\nu} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i}^{\nu} b^i(x) \frac{\partial}{\partial x_i}$$

where the matrix $(a^{ij}(x))$ is positive definite for each $x$ from some neighborhood of $G$. In fact, everything goes through if we consider $L + c(x)$ for a continuous function $c$, i.e. when $L^\varepsilon = \varepsilon L + \varepsilon c + (B, \nabla) + V$. Indeed, substituting $(V + \varepsilon c)$ in place of $V$ into (1.6) we see by (1.5) that

$$|\lambda_G^\varepsilon(V) - \lambda_G^\varepsilon(V + \varepsilon c)| \leqq \varepsilon \sup_{x \in G} |c(x)|,$$

and so as $\varepsilon \to 0$ we have the same asymptotics.

In order to obtain more interesting results one needs some assumptions about the structure of $F^t$-invariant sets in $G$. As in [Ki4] we will need the notion of $\delta$-pseudo-orbits which are sequences of points $x_0, \ldots, x_n \in G$ satisfying

$$\text{(2.2)} \qquad \operatorname{dist}(F^1 x_i, x_{i+1}) \leqq \delta \quad \text{for } i = 0, 1, \ldots, n-1.$$

For a pair of points $x, y \in \bar{G}$ we write $x \to y$ if for any $\delta > 0$ there exist a non-negative $t \leqq 1$ and a $\delta$-pseudo-orbit $x_0, \ldots, x_n \in \bar{G}$ such that $F^t x = x_0$ and $x_n = y$. We extend this relation to a transitive one " $\prec$ " so that $x \prec z$ iff there exists a sequence of points $y_0, \ldots, y_k \in \bar{G}$ such that $y_0 = x$, $y_k = z$, and $y_i \to y_{i+1}$ for all $i = 0, 1, \ldots, k-1$. If $x \prec y$ and $y \prec x$ we write $x \sim y$ and call $x$ and $y$ equivalent points. As usual, any maximal set of equivalent points in $\bar{G}$ will be called an equivalence class which in our case must be a closed set.

A closed set $K$ is called $F^t$-invariant if $F^t K = K$ for all $t$. Suppose that there exists a finite collection of $F^t$-invariant equivalence classes $K_1, \ldots, K_\kappa \subset G \cup \partial G$ satisfying the following Assumption A:

(A1) $\bigcup_i K_i$ contains the limit set of the dynamical system $F^t$ in $\bar{G}$, i.e. for any $x \in \bar{G}$ all limit points of $F^t x$ as $t \to \pm\infty$ which belong to $\bar{G}$ belong also to $\bigcup_i K_i$;

(A2) one can choose open disjoint sets $U_i \subset M$, $i = 1, \dots, \kappa$ with piecewise smooth boundaries $\partial U_i$ such that $U_i \supset K_i$, the limit

$$\lim_{\varepsilon \to 0} \lambda^{\varepsilon}_{U_i \cap G}(V) = \lambda_{K_i}(V)$$

exists, and for some positive $\beta_0 < 1$ and each $\delta > 0$ there is $\varepsilon(\delta) > 0$ so that if $\varepsilon \leqq \varepsilon(\delta)$ then one can find a positive $t(\varepsilon, \delta) \leqq \varepsilon^{-2(1-\beta_0)}$ satisfying

$$\| T^{\varepsilon}_{V, U_i \cap G}(t(\varepsilon, \delta)) \| \leqq \exp((\lambda_{K_i}(V) + \delta) t(\varepsilon, \delta)). \tag{2.3}$$

The following result generalizes Theorem 2.1 from [EK] where $V \equiv 0$ was assumed. Recall that the case $G = M$ with $M$ compact is included, as well.

Theorem 2.1. *Under Assumption* A

$$\lim_{\varepsilon \to 0} \lambda^{\varepsilon}_G(V) = \max_{1 \leqq i \leqq \kappa} \lambda_{K_i}(V) \tag{2.4}$$

*and the numbers* $\lambda_{K_i}(V)$ *defined by* (2.2) *are determined by compacts* $K_i$ *only, i.e. they do not depend on the choice of* $U_i$.

This theorem enables one to reduce the problem to the study of principal eigenvalues of operators $L^{\varepsilon}_V$ restricted to small neighborhoods of compacts $K_i$. This can be accomplished for certain types of $F^t$-invariant sets $K_i$. A glance at the Feynman–Kac formula (1.2) shows that the limiting behavior in (1.4) involves a fight between the functional $\exp(\int_0^t V(X^{\varepsilon}_s) ds)$ and the exponential rate of decay of the probability for the process $X^{\varepsilon}_s$ to stay in $G$ (if $G$ is not the whole $M$) up to the time $t$.

We will start with hyperbolic invariant sets. A compact $F^t$-invariant set $K \subset M$ is called hyperbolic if the tangent boundle $TM$ restricted to $K$ can be written as the Whitney sum of continuous subbundles $T_K M = \Gamma^s \oplus \Gamma^u$, where $\Gamma^0$ is the one-dimensional bundle tangent to the flow $F^t$, this decomposition is invariant with respect to the differential $DF^t$ of $F^t$, and there exist constants $C_1, \alpha_0 > 0$ such that

$$\| DF^t \xi \| \leqq C_1 e^{-\alpha_0 t} \| \xi \| \quad \text{for } \xi \in \Gamma^s, \quad t \geqq 0$$

and

$$\| DF^{-t} \zeta \| \leqq C_1 e^{-\alpha_0 t} \| \zeta \| \quad \text{for } \zeta \in \Gamma^u, \quad t \geqq 0.$$

A hyperbolic set $K$ is called basic hyperbolic if $K$ is a fixed point or contains no fixed points, the periodic orbits of $F^t|_K$ are dense in $K$, $F^t|_K$ is topologically transitive, and there is an open set $U \supset K$ with $K = \bigcap_{-\infty<t<\infty} F^tU$.

The topological pressure $P_K(F, \psi)$ of the flow $F^t$ on a $F^t$-invariant set $K$ given a continuous function $\psi$ was defined by (1.19). Let $K$ be a basic hyperbolic set and $x \in K$. Denote by $J_t(x)$ the Jacobian of the linear map $DF^t : \Gamma^u_x \to \Gamma^u_{F^tx}$ with respect to inner products induced by the Riemannian metric. Define

$$\varphi^u(x) = -\left.\frac{dJ_t(x)}{dt}\right|_{t=0}. \tag{2.5}$$

I proved in [Ki2] (see also [Ki4]) that in the above circumstances $\lambda_K(0) = P_K(F, \varphi^u)$. We will obtain the following generalization:

THEOREM 2.2. *If $K$ is a basic hyperbolic set then the limit*

$$\lambda_K(V) = \lim_{\varepsilon\to 0} \lambda^\varepsilon_U(V) = P_K(F, \varphi^u + V) \tag{2.6}$$

*exists, it is independent of an open set $U \supset K$, $U \subset G$ provided $\bigcap_{-\infty<t<\infty} F^t\bar{U} = K$ (i.e. there is no other $F^t$-invariant set in $\bar{U}$), and a corresponding version of* (2.3) *holds true, as well.*

In the partial case when $K$ is a hyperbolic fixed point of $F^t$ we obtain that $\lambda_K(V) = V(K) - \Sigma_i \max(\operatorname{Re}\delta_i, 0)$ where $\delta_i$, $i = 1, \ldots, \mu$ are all eigenvalues of the matrix $\Pi$ such that $B(x) = \Pi(x - K) + 0(|x - K|^2)$.

We remark that if Assumption A holds true and all $K_i$ are basic hyperbolic sets contained in $G$ then the right hand side of (2.4) can be written in the form $P_{\cup_i K_i}(F, \varphi^u + V)$ where $\varphi^u$ is defined by (2.6) on each of $K_i$. Indeed, all $F^t$-invariant probability measures in $\bar{G}$ have the form $\mu = \Sigma_{1\leqq i\leqq \kappa} p_i\mu_i$, where $p_i \geqq 0$, $\Sigma_i p_i = 1$, and $\operatorname{supp}\mu_i \subset K_i$. Since the entropy $h_\mu(F^1)$ is affine in $\mu$ (see [W]) then

$$h_\mu(F^1) + \int (\varphi^u + V)d\mu = \sum_i p_i\left(h_{\mu_i}(F^1) + \int (\varphi^u + V)d\mu\right) \tag{2.7}$$

and so $P_{\cup_i K_i}(F, \varphi^u + V) = \max_i P_{K_i}(F, \varphi^u + V)$. Hence if $F^t$ is an Axiom-A flow on a compact $M$ then

$$\lim_{\varepsilon\to 0} \lambda^\varepsilon_M(V) = P_M(F, \varphi^u + V). \tag{2.8}$$

The topological pressure given by the formula (1.19) makes sense not only for hyperbolic sets. It would be interesting to obtain (2.8) for other classes of

$F^t$-invariant sets. In more general circumstances when we cannot define $\varphi^u$ by (2.5) it can be still obtained sometimes via the limit

$$(2.9) \qquad \varphi^u(x) = \lim_{t\to\infty} \frac{1}{t} \ln \operatorname{vol}\{y : \operatorname{dist}(F^s x, F^s y) \leqq \delta \text{ for all } s \in [0, t]\}$$

which often exists and is independent of $\delta > 0$ small enough. Besides $\varphi^u$ appears in $P_K(F, \varphi^u + V)$ only via integrals $\int \varphi^u d\mu$ with respect to $F^t$-invariant measures $\mu$ which make perfect sense for all smooth dynamical systems as minus integral of the sum of positive Lyapunov characteristic exponents for $\mu$.

In the above case $K$ was supposed to be strictly inside of $G$. Assume now that $K$ is an $F^t$-invariant connected component $\Gamma$ of the boundary $\partial G$. Thus $\Gamma$ is a closed smooth surface of codimension one. It is easy to see that one can pick up an open neighborhood $U$ of $\Gamma$ in $M$ such that any point $x \in U$ has a unique representation $x = \gamma(x) + \rho(x)n(x)$ where $\gamma(x) \in \Gamma$, $|\rho(x)| = |x - \gamma(x)| = \operatorname{dist}(x, \Gamma)$, and $n(x) = n(\gamma(x))$ is the interior unit normal to $\Gamma$ in the sense that it points out into the interior of $G$, i.e. $\rho(x) > 0$ if $x \in U \cap G$. Characterizing any point $x \in U$ by the pair $(\gamma(x), \rho(x))$ we get a system of coordinates in $U$. In these coordinates the normal component $q(x)$ of the vector field $B(x)$ satisfies

$$\frac{d\rho(F^t x)}{dt} = q(F^t x) = q(\gamma(F^t x), \rho(F^t x)).$$

For each $\gamma \in K$ define

$$\beta(\gamma) = \left.\frac{\partial q(\gamma, \rho)}{\partial \rho}\right|_{\rho=0}$$

and assume that uniformly in $\gamma \in K$ the limits

$$(2.10) \qquad \lim_{t\to\infty} \frac{1}{t} \int_0^t \beta(F^s \gamma) ds = \beta_0 \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{t} \int_0^t V(F^s \gamma) ds = v_0$$

exist.

THEOREM 2.3. *In the above circumstances let $U$ be an open neighborhood of $\Gamma$ with a smooth boundary $\partial U$ such that $U \cup \partial U$ contains no closed $F^t$-invariant sets except for $\Gamma$. If $\beta_0 < 0$ then*

$$\lambda_\Gamma(V) = \lim_{\varepsilon \to 0} \lambda^\varepsilon_{U \cap G}(V) = \beta_0 + v_0, \tag{2.11}$$

*and if* $\beta_0 > 0$, *and, in addition, the dynamical system* $S^t$ *restricted to* $\Gamma$ *has an invariant measure on* $\Gamma$ *possessing a smooth positive density with respect to the volume, then*

$$\lambda_\Gamma(V) = \lim_{\varepsilon \to 0} \lambda^\varepsilon_{U \cap G}(V) = -2\beta_0 + v_0. \tag{2.12}$$

*In both cases the condition* (2.3) *is satisfied.*

The proof of this result is essentially the repetition of the proof of Theorem 2.2 from [EK] and we will not give it here.

The above results enable one to estimate decay and growth rates of solutions for the quasi-linear equation (1.11).

PROPOSITION 2.4. (i) *Let* $w^\varepsilon_g(t, x)$ *be a solution of* (1.11) *and the condition* (1.13) *be satisfied. If uniformly in* $x$,

$$\lim_{t \to \infty} w^\varepsilon_g(t, x) = 0 \tag{2.13}$$

*then* (1.15) *follows and if* $g \geqq 0$, $g \not\equiv 0$ *then*

$$\lim_{t \to \infty} \frac{1}{t} \ln w^\varepsilon_g(t, x) = \lambda^\varepsilon_G(V_0). \tag{2.14}$$

*The condition* (2.13) *is satisfied for all bounded* $g$ *if, for instance,* (1.14) *holds true.*

(ii) *Suppose that uniformly in* $x \in G$ *the limit*

$$V_\infty(x) = \lim_{u \to \infty} V(x, u) \tag{2.15}$$

*exists. Let* $w^\varepsilon_g(t, x)$ *be a solution of* (1.11) *such that uniformly in* $x$

$$\lim_{t \to \infty} w^\varepsilon_g(t, x) = \infty \tag{2.16}$$

*then* (1.15) *holds true with* $V_0$ *replaced by* $V_\infty$. *If* $g \geqq 0$, $g \not\equiv 0$ *then*

$$\lim_{t \to \infty} \frac{1}{t} \ln w^\varepsilon_g(t, x) = \lambda^\varepsilon_G(V_\infty). \tag{2.17}$$

*The condition* (2.15) *is satisfied for all bounded* $g \geqq 0$, $g \not\equiv 0$ *if, for instance,*

$$\inf_{x \in M, u} V(x,u) > -\lambda_G^\varepsilon(0). \tag{2.18}$$

REMARK 2.1. All results of this paper go through if the coefficients of $L$ and $B$ are only $C^1$ but have Hölder continuous derivatives.

REMARK 2.2. Under conditions of Theorem 2.1–2.3 the number $\lambda_G^\varepsilon(0)$ can be estimated for small $\varepsilon$ and so the condition (1.14) and (2.18) can be verified. The same theorems specify the behavior as $\varepsilon \to 0$ of the right hand sides $\lambda_G^\varepsilon(V_0)$ and $\lambda_G^\varepsilon(V_\infty)$ of (2.14) and (2.17), as well.

REMARK 2.3. Results similar to Theorem 2.2 can be obtained also when compacts $K_i$ turn out to be certain types of normally hyperbolic manifolds, say, when we have on $K_i$ a flow diffeomorphically conjugate to an irrational rotation on a torus with a hyperbolic structure in the transversal to $K_i$ subbundle. Then a modification of the proof of Theorem 2.2 from [EK] will provide the limit $\lambda_{K_i}(V)$ (cf. Remark 7.1 in [EK]).

REMARK 2.4. The same method works when $G = M$ is not necessarily compact, say $G = M = R^n$, but then the vector field $B$ points inside, so that the flow $F^t$ attracts every point to a compact domain containing all $F^t$-invariant subsets in the spirit of Theorem 1.7, Chapter I from [Ki4]. Then the process $X_t^\varepsilon$ becomes strongly recurrent and we can study the behavior of principal eigenvalues as $\varepsilon \to 0$ in the same way as above.

## 3. Consequences of variational formulas

Supose that $\mathscr{L}$ is an arbitrary second order strongly elliptic differential operator on $M$ with $C^2$ coefficients and $\mathscr{L}1 = 0$. For any continuous function $V$ on $M$ denote by $\Sigma_{V,G}^{\mathscr{L}}$ the spectrum of the operator $\mathscr{L}_V = \mathscr{L} + V$ corresponding to the Dirichlet boundary conditions on $\partial G$ if the domain $G$ as above is not the whole $M$ and without boundary conditions if $G = M$ and $M$ is compact. Let $\lambda_G^{\mathscr{L}}(V) = \sup\{\mathrm{Re}\,\mathscr{L} : \lambda \in \Sigma_{V,G}^{\mathscr{L}}\}$ then by [DV1] and [DV3], $\lambda_G^{\mathscr{L}}(V) \in \Sigma_{V,G}^{\mathscr{L}}$ and one has the representation

$$\lambda_G^{\mathscr{L}}(V) = \sup_{\mu \in \mathscr{P}(G)} \left( \int V d\mu - I_G^{\mathscr{L}}(\mu) \right) \tag{3.1}$$

where

$$I_G^{\mathscr{L}}(\mu) = -\sup_{u \in D_+^{\mathscr{L}}} \int \frac{\mathscr{L}u}{u} d\mu \tag{3.2}$$

and $\mathscr{D}_{+}^{\mathscr{L}}$ is the set of functions from the domain of $\mathscr{L}$ having positive lower bounds.

If $V$ is Hölder continuous then it follows from the general theory of positive operators (see §2 Ch. 7 of [Kr]) that $\lambda_G^{\mathscr{L}}(V)$ is an eigenvalue of $\mathscr{L}_V$ called the principal eigenvalue. The corresponding eigenfunction $r_{V,G}^{\mathscr{L}}$ is called the principal eigenfunction. It is unique and $r_{V,G}^{\mathscr{L}}$ is the only positive eigenfunction of $\mathscr{L}_V$.

Consider the Markov semigroup of operators $\mathscr{T}_{V,G}^{\mathscr{L}}(t)$ acting by the formula

$$\mathscr{T}_{V,G}^{\mathscr{L}}(t)g(x) = e^{-\lambda_G^{\mathscr{L}}(V)t}(r_{V,G}^{\mathscr{L}}(x))^{-1}E_x(\chi_{\tau>t}g(\mathscr{X}_t)r_{V,G}^{\mathscr{L}}(\mathscr{X}_t)e^{\int_0^t V(\mathscr{X}_s)ds}) \tag{3.3}$$

where $\mathscr{X}_t$ is the diffusion process generated by the operator $\mathscr{L}$ and $\tau$ is its first exit time from $G$ if $G \neq M$ and $\tau = \infty$ if $G = M$. Clearly,

$$\mathscr{A} = \frac{1}{r_{V,G}^{\mathscr{L}}}(\mathscr{L}_V - \lambda_G^{\mathscr{L}}(V))r_{V,G}^{\mathscr{L}} \tag{3.4}$$

is the generator of the semigroup $\mathscr{T}_{V,G}^{\mathscr{L}}(t)$. The operator $\mathscr{A}$ generates a Markov process $\mathscr{Y}_t^V$ with transition probabilities

$$P_V^{\mathscr{L}}(t, x, \Gamma) = \mathscr{T}_{V,G}^{\mathscr{L}}(t)\chi_\Gamma(x).$$

Since $G$ will be fixed in this section we will usually omit the index $G$.

PROPOSITION 3.1. *Let $V$ be Hölder continuous then there exists a unique probability measure $\mu_V^{\mathscr{L}}$ on $\bar{G}$ such that*

$$\lambda^{\mathscr{L}}(V) = \int V d\mu_V^{\mathscr{L}} - I^{\mathscr{L}}(\mu_V^{\mathscr{L}}) \tag{3.5}$$

*and this is the only invariant probability measure of the Markov process $\mathscr{Y}_t^V$. This measure has a smooth density $\pi_V^{\mathscr{L}}$ with respect to the Riemannian volume $m$ on $M$, $\pi_V^{\mathscr{L}}(x) \to 0$ as $x \to \partial G$, and the transition density $p_V^{\mathscr{L}}(t, x, y)$ of $\mathscr{Y}_t^V$ converges exponentially fast to $\pi_V^{\mathscr{L}}(y)$ as $t \to \infty$.*

PROOF. Let $q_V^{\mathscr{L}}(t, x, y)$ be the Green's function (see [F1]) of the parabolic equation $\partial u/\partial t = \mathscr{L}_V u$ corresponding to zero data on $\partial G$ (if $G \neq M$). Then by (3.3),

$$p_V^{\mathscr{L}}(t, x, y) = e^{-\lambda^{\mathscr{L}}(V)t}(r_V^{\mathscr{L}}(x))^{-1}q_V^{\mathscr{L}}(t, x, y)r_V^{\mathscr{L}}(y) \tag{3.6}$$

is the transition density of the Markov process $\mathscr{Y}_t^V$. Let $r_V^{\mathscr{L}*}$ be the eigenfunction of the adjoint operator $\mathscr{L}_V^* = \mathscr{L}^* + V$ corresponding to the same eigenvalue $\lambda^{\mathscr{L}}(V)$ normalized by $\int r_V^{\mathscr{L}} r_V^{\mathscr{L}*} dm = 1$. Then $\pi_V^{\mathscr{L}} = r_V^{\mathscr{L}} r_V^{\mathscr{L}*}$ is the density of the invariant measure of the process $\mathscr{Y}_t^V$ since

$$\int_{\bar G} dm(x)\pi_V^{\mathscr{L}}(x)\, p_V^{\mathscr{L}}(t,x,y) = e^{-\lambda^{\mathscr{L}}(V)t}\int_{\bar G} dm(x) r_V^{\mathscr{L}*}(x) q_V^{\mathscr{L}}(t,x,y) r_V^{\mathscr{L}}(y)$$

$$= e^{-\lambda^{\mathscr{L}}(V)t}\int_{\bar G} q_V^{\mathscr{L}*}(t,y,x) r_V^{\mathscr{L}*}(x) dm(x) r_V^{\mathscr{L}}(y)$$

$$= r_V^{\mathscr{L}}(y) r_V^{\mathscr{L}*}(y)$$

where $q_V^{\mathscr{L}*}$ is the Green's function of the adjoint equation. Thus the probability measure $\mu_V^{\mathscr{L}}$ defined by

$$\mu_V^{\mathscr{L}}(\Gamma) = \int_\Gamma \pi_V^{\mathscr{L}}(x) dm(x)$$

for any Borel set $\Gamma \subset \bar G$ is the invariant measure of the process $\mathscr{Y}_t^V$. At this point we remark that when $V \equiv 0$ the process $\mathscr{Y}_t^V$ and its invariant measure $\mu_V^{\mathscr{L}}$ were studied in [P] and [GQZ]. In the last paper the uniqueness of $\mu_V^{\mathscr{L}}$ for $V \equiv 0$ and its mixing properties were established. We will give here another argument based on Doeblin's condition.

Since both $e^{\lambda^{\mathscr{L}}(V)t} r_V^{\mathscr{L}}(x)$ and $q_V^{\mathscr{L}}(t,x,y)$ as functions of $(t,x)$ are positive solutions of the equation $\partial u/\partial t = \mathscr{L}_V u$ and $q_V^{\mathscr{L}}(t,x,y)$ as a function of $(t,y)$ is also a solution of the adjoint equation then by the strong maximum principle the outward normal derivatives along $\partial G$ of $r_V^{\mathscr{L}}$ in $x$ and of $q_V^{\mathscr{L}}$ in $x$ and in $y$ are strictly negative. Since these derivatives are continuous we derive from (3.6) that $p_V^{\mathscr{L}}(t,x,y)$ can be continuously extended into the whole $\bar G$ and this extended transition density denoted again $p_V^{\mathscr{L}}$ satisfies $p_V^{\mathscr{L}}(t,x,y) = 0$ if $y \in \partial G$ and for each $t, \delta > 0$,

$$\inf_{x \in (\bar G)} p_V^{\mathscr{L}}(t,x,y) \geqq c_{t,\delta} \qquad \text{provided } \operatorname{dist}(y, \partial G) \geqq \delta \tag{3.7}$$

for some $c_{t,\delta} > 0$. Of course, none of these is needed if $G = M$ is compact since then $r_V^{\mathscr{L}}$ and $q_V^{\mathscr{L}}(t,x,y)$ are positive smooth functions on $M$ and in this case one obtains for each $t > 0$ that

$$\inf_{x,y \in M} p_V^{\mathscr{L}}(t,x,y) > 0. \tag{3.8}$$

Now we are under Doeblin's condition (see, for instance, [Ki4]) which implies the existence and uniqueness of the invariant measure for $\mathscr{Y}_t^V$ and, moreover (see [BK], Proposition 2.8), (3.9) $$|p_V^{\mathscr{L}}(t,x,y)-\pi_V^{\mathscr{L}}(y)|\leqq Ce^{-\gamma t}$$

for some $C,\gamma>0$.

Next, denote for any $\mu\in\mathscr{P}(\bar{G})$,

(3.10) $$I^{\mathscr{A}}(\mu)=-\inf_{u\in\mathscr{D}_+^{\mathscr{L}}}\int_{\bar{G}}\frac{\mathscr{A}(u/r_V^{\mathscr{L}})}{(u/r_V^{\mathscr{L}})}\,d\mu,$$

then by (3.4),

(3.11) $$I^{\mathscr{A}}(\mu)=I^{\mathscr{L}}(\mu)-\int Vd\mu+\lambda^{\mathscr{L}}(V).$$

Thus it remains to show that $\mu$ is an invariant measure of $\mathscr{Y}_t^V$ if and only if $I^{\mathscr{A}}(\mu)=0$. To do this one has to mimic Lemmas 2.5 and 3.1 from [DV2]. This is straightforward if $G=M$ is compact since then $r_V^{\mathscr{L}}$ is a $C^2$ positive function on $M$, and so it is bounded away from zero. If $\partial G$ is not empty then $r_V^{\mathscr{L}}(x)\to 0$ as $x\to\partial G$ and so some justification is needed.

For $\mu\in\mathscr{P}(\bar{G})$ denote

(3.12) $$\mathscr{I}_t(\mu)=-\inf_{u\in C_+(\bar{G})}\int_{\bar{G}}\ln\left(\frac{\mathscr{T}_V^{\mathscr{L}}(t)u}{u}\right)d\mu$$

and

(3.13) $$\tilde{\mathscr{I}}_t(\mu)=-\inf_{u\in C_+(\bar{G})}\int_{\bar{G}}\ln\left(\frac{\mathscr{T}_V^{\mathscr{L}}(t)(u/r_V^{\mathscr{L}})}{(u/r_V^{\mathscr{L}})}\right)d\mu$$

where $C_+(\bar{G})$ is the space of positive continuous functions on $\bar{G}$. As we have seen above, the transition operator $\mathscr{T}_V^{\mathscr{L}}(t)$ of the process $\mathscr{Y}_t^V$ has the continuous kernel $p_V^{\mathscr{L}}(t,x,y)$, and so one can apply Lemma 2.5 of [DV2] to conclude that $\mathscr{I}_t(\mu)=0$ if and only if $\mu$ is the invariant measure of the process $\mathscr{Y}_t^V$. I claim that $\mathscr{I}_t(\mu)=\tilde{\mathscr{I}}_t(\mu)$. Indeed, take a sequence $u_n\in C_+(\bar{G})$ such that $u_n\downarrow r_V^{\mathscr{L}}$ as $n\uparrow\infty$ then for any $u\in C_+(\bar{G})$ we have $u/u_n\in C_+(\bar{G})$ and $u/u_n\uparrow u/r_V^{\mathscr{L}}$ as $n\uparrow\infty$. Then,

$$\mathscr{I}_t(\mu)=-\lim_{n\to\infty}\inf_{u\in C_+(\bar{G})}\int_{\bar{G}}\ln\left(\frac{\mathscr{T}_V^{\mathscr{L}}(t)(u/u_n)}{(u/u_n)}\right)d\mu$$

(3.14) $$\geqq-\inf_{u\in C_+(\bar{G})}\lim_{n\to\infty}\int_{\bar{G}}\ln\left(\frac{\mathscr{T}_V^{\mathscr{L}}(t)(u/u_n)}{(u/u_n)}\right)d\mu$$

$$=\tilde{\mathscr{I}}_t(\mu).$$

The last equality in (3.14) holds true in view of the formula (3.6) for the kernel $p_V^{\mathscr{L}}(t,x,y)$ of the operator $\mathscr{T}_V^{\mathscr{L}}(t)$ and the strong maximum principle which gives the positivity of inward normal derivatives on $\partial G$ of the Green's function $q_V^{\mathscr{L}}$ and the eigenfunction $r_V^{\mathscr{L}}$. Similarly,

$$\begin{aligned} \tilde{\mathscr{I}}_t(\mu) &= -\lim_{n\to\infty}\inf_{u\in C_+(\bar{G})}\int_G \ln\left(\frac{\mathscr{T}_V^{\mathscr{L}}(t)(uu_n/r_V^{\mathscr{L}})}{(uu_n/r_V^{\mathscr{L}})}\right)d\mu \\ &\geqq -\inf_{u\in C_+(\bar{G})}\lim_{n\to\infty}\int_G \ln\left(\frac{\mathscr{T}_V^{\mathscr{L}}(t)(uu_n/r_V^{\mathscr{L}})}{(uu_n/r_V^{\mathscr{L}})}\right)d\mu \qquad (3.15)\\ &= \mathscr{I}_t(\mu). \end{aligned}$$

Next, in the same way as in Lemma 3.1 of [DV2] one obtains that for any $t>0$,

$$\tilde{\mathscr{I}}_t(\mu)\leqq tI^{\mathscr{A}}(\mu) \quad \text{and} \quad \lim_{t\to\infty}\frac{1}{t}\tilde{\mathscr{I}}_t(\mu)=I^{\mathscr{A}}(\mu). \tag{3.16}$$

This together with the arguments above says that $I^{\mathscr{A}}(\mu)=0$ if and only if $\mu$ is an invariant measure of $\mathscr{Y}_t^V$ which, together with (3.11) and the uniqueness of an invariant measure, completes the proof of Proposition 3.1. □

The following two results were indicated to me by S. R. S. Varadhan though the next statement seems to be standard in convex analysis.

Proposition 3.2. *Let $J^{(i)}(\mu)$, $i=0,1,2,\dots$ be convex lower semi-continuous nonnegative functionals on the space $\mathscr{P}(Y)$ of probability measures on a compact space $Y$. For any $V$ from the space $C(Y)$ of continuous functions on $Y$ and $i=0,1,2,\dots$ set*

$$\lambda^{(i)}(V)=\sup_{\mu\in\mathscr{P}(Y)}\left(\int_Y Vd\mu - J^{(i)}(\mu)\right) \tag{3.17}$$

*and*

$$\mathscr{N}_V^{(i)}=\left\{\mu\in\mathscr{P}(Y):\lambda^{(i)}(V)=\int_Y Vd\mu-J^{(i)}(\mu)\right\}.$$

*Assume that*

$$\lim_{t\to\infty}\lambda^{(i)}(V)=\lambda^{(0)}(V) \quad \textit{for all } V\in C(Y). \tag{3.18}$$

*Then all limit points as $i\to\infty$ of any sequence of measure $\mu^{(i)}\in\mathscr{N}_V^{(i)}$ belong to $\mathscr{N}_V^{(0)}$.*

PROOF. First, we remark that each $\mathcal{N}_V^{(i)}$ is nonempty. Indeed, take a sequence $\mu_k \in \mathcal{P}(Y)$ such that

$$\lambda^{(i)}(V) = \lim_{k\to\infty}\left(\int_Y V d\mu_k - J^{(i)}(\mu_k)\right). \tag{3.19}$$

Since $Y$ is compact we can choose a weakly convergent subsequence which we denote again by $\mu_k$, $\mu_k \xrightarrow{w} \mu$. Then

$$\liminf_{k\to\infty} J^{(i)}(\mu_k) \geqq J^{(i)}(\mu)$$

by the lower semicontinuity of $J^{(i)}(\mu)$, and so by (3.17) and (3.19), $\mu \in \mathcal{N}_V^{(i)}$.

Next, suppose that $\mu_V^{(i)} \in \mathcal{N}_N^{(i)}$, then

$$\lambda^{(i)}(W) \geqq \int_Y W d\mu_V^{(i)} - J^{(i)}(\mu_V^{(i)})$$

for any continuous $W$, and so

$$\lambda^{(i)}(W) - \lambda^{(i)}(V) \geqq \int_Y (W - V) d\mu_V^{(i)}. \tag{3.20}$$

Assume that $\mu_V^{(i)} \xrightarrow{w} \mu$ as $i \to \infty$, then by (3.18) and (3.20),

$$\int_Y V d\mu - \lambda^{(0)}(V) \geqq \int_Y W d\mu - \lambda^{(0)}(W)$$

for all $W \in C(Y)$, and so

$$\int_Y V d\mu - \lambda^{(0)}(V) = \sup_{W\in C(Y)}\left(\int_Y W d\mu - \lambda^{(0)}(W)\right). \tag{3.21}$$

Since $J^{(0)}$ is a convex lower semicontinuous functional (remark that we do not use convexity of $J^{(i)}$ for $i \neq 0$) then the well known duality of convex functionals yields from (3.17) that

$$J^{(0)}(\mu) = \sup_{W\in C(Y)}\left(\int_Y W d\mu - \lambda^{(0)}(W)\right) \tag{3.22}$$

for any $\mu \in \mathcal{P}(Y)$. For the reader's convenience we will show (3.22) mimicking the proof of Theorem 9.12 in [W]. Since by (3.17), $\lambda^{(0)}(W) \geqq \int W d\mu - J^{(0)}(\mu)$ for all $W \in C(Y)$ and $\mu \in \mathcal{P}(Y)$, then $J^{(0)}(\mu) \geqq \int W d\mu - \lambda^{(0)}(W)$ and so

$$J^{(0)}(\mu) \geqq \sup_{W\in C(Y)}\left(\int_Y W d\mu - \lambda^{(0)}(W)\right). \tag{3.23}$$

To prove the opposite inequality fix $\mu \in \mathscr{P}(Y)$ and take an arbitrary number $b < J^{(0)}(\mu)$. Let $K = \{(\nu, t) : \nu \in \mathscr{P}(Y),\ J^{(0)}(\mu) \leqq t \leqq \infty\}$. Since $J^{(0)}(\mu)$ is a convex functional, i.e. for any $p_1, p_2 \geqq 0$, $p_1 + p_2 = 1$,

$$J^{(0)}(p_1\mu_1 + p_2\mu_2) \leqq p_1 J^{(0)}(\mu_1) + p_2 J^{(0)}(\mu_2),$$

then $K$ is a convex set which is also closed by the lower semicontinuity of $J^{(0)}$. Since $(\mu, b) \notin K$ we can use the separation theorem for convex sets on p. 417 of [DS] saying that there exists a continuous linear functional $l$ such that $l(\mu, b) < l(\nu, t)$ for all $(\mu, t) \in K$. By a version of the Riesz representation theorem (see [DS], Section IV, 6), $l(\nu, t) = \int W d\nu - rt$ for some $W \in C(Y)$ and a number $r$. Hence $\int W d\mu - rb < \int W d\nu - rt$ for all $(\nu, t) \in K$. In particular, $(\nu, J^{(0)}(\nu)) \in K$, and so $\int W d\mu - rb < \int W d\nu - rJ^{(0)}(\nu)$ for any $\nu \in \mathscr{P}(Y)$. Taking $\nu = \mu$ we conclude that $r < 0$. Put $\tilde{W} = W/r$ then $\int \tilde{W} d\mu - b > \int \tilde{W} d\nu - J^{(0)}(\nu)$ for any $\nu \in \mathscr{P}(Y)$. Thus by (3.17), $\int \tilde{W} d\mu - b \geqq \lambda^{(0)}(\tilde{W})$, i.e. $\int \tilde{W} d\mu - \lambda^{(0)}(\tilde{W}) \geqq b$, and so $\sup_{W \in C(Y)}(\int W d\mu - \lambda^{(0)}(W)) \geqq b$. Since $b$ can be any number less than $J^{(0)}(\mu)$ we derive $\sup_{W \in C(Y)}(\int W d\mu - \lambda^{(0)}(W)) \geqq J^{(0)}(\mu)$ which together with (3.23) gives (3.22).

Finally, (3.21) and (3.22) yield

$$\int_Y V d\mu - \lambda^{(0)}(V) = J^{(0)}(\mu), \quad \text{and so } \mu \in \mathscr{N}_V^{(0)}. \qquad \square$$

Proposition 3.3. *Let $I_G^\varepsilon(\mu)$ be defined by* (1.17). *Suppose that for some sequence of measures $\mu_i \in \mathscr{P}(\bar{G})$ and a sequence of numbers $\varepsilon_i \to 0$ there is a constant $C < \infty$ such that*

$$I_G^{\varepsilon_i}(\mu_i) \leqq C \quad \textit{for all } i = 1, 2, \ldots . \tag{3.24}$$

*Then any weak limit as $i \to \infty$ of measure $\mu_i$ will be an invariant measure of the flow $F^t$ generated by* (1.1), *and so the set $\mathscr{M}_G^F$ of $F^t$-invariant probability measures with support in $\bar{G}$ is not empty or, equivalently, there exists an $F^t$-invariant set in $\bar{G}$. If, on the other hand, $\mathscr{M}_G^F$ is empty then*

$$I_G^{\varepsilon_i}(\mu_i) \to \infty \quad \textit{as } i \to \infty \tag{3.25}$$

*for any sequence $\varepsilon_i \to 0$ and $\mu_i \in \mathscr{P}(\bar{G})$. In particular, if $\mu_V^\varepsilon$ is defined by* (*uniquely, by Proposition* 3.1 *if $V$ is Hölder continuous*)

$$\lambda_G^\varepsilon(V) = \int_G V d\mu_V^\varepsilon - I_G^\varepsilon(\mu_V^\varepsilon) \tag{3.26}$$

*and $\mathcal{M}_G^F$ is not empty then any weak limit as $\varepsilon \to 0$ of measures $\mu_V^\varepsilon$ belong to $\mathcal{M}_G^F$.*

PROOF. Let $C_+^2(\bar{G})$ be the set of $u \in D_+$ which are $C^2$ in a neighborhood of $\bar{G}$. It is known (see [DV3]) that the infinum in (1.17) can be taken over $u \in C_+^2(\bar{G})$ only. For such $u$ we have $\varepsilon_i Lu \to 0$ as $\varepsilon_i \to 0$ uniformly in $\bar{G}$. Thus if $\mu_i \xrightarrow{w} \mu$ as $i \to \infty$ and $I_G^{\varepsilon_i}(\mu_i) \leqq C$ then

$$
\begin{aligned}
C &\geqq \liminf_{i\to\infty} I_G^{\varepsilon_i}(\mu_i) = \liminf_{i\to\infty}\left( -\inf_{u\in C_+^2(\bar{G})} \int_{\bar{G}} \frac{L^{\varepsilon_i}u}{u}\, d\mu_i \right) \\
&\geqq -\inf_{u\in C_+^2(\bar{G})} \limsup_{i\to\infty} \int_{\bar{G}} \frac{L^{\varepsilon_i}u}{u}\, d\mu_i \qquad (3.27)\\
&= -\inf_{u\in C_+^2(\bar{G})} \int_{\bar{G}} \frac{Bu}{u}\, d\mu \overset{\text{def}}{=} I_G^B(\mu).
\end{aligned}
$$

Since $B$ is the first order operator, and so $Bu^n = nu^{n-1}Bu$, then

$$
I_G^B(\mu) \geqq -\inf_{u\in C_+^2(\bar{G})} \int_{\bar{G}} \frac{B(u^n)}{u^n}\, d\mu = nI_G^B(\mu). \tag{3.28}
$$

As usual, $I_G^B(\mu) \geqq 0$ since $u \equiv 1$ gives zero in the above integral, and so (3.28) leaves only two choices $I_G^B(\mu) = 0$ or $= \infty$. This together with (3.27) yields $I_G^B(\mu) = 0$. Again we repeat the argument from Lemma 2.5 of [DV2]. Since we have

$$
\int \frac{Bu}{u}\, d\mu \geqq 0 \quad \text{for any } u \in C_+^2(\bar{G})
$$

then

$$
\int \frac{B(1+\varepsilon u)}{1+\varepsilon u}\, d\mu
$$

attains its minimum at $\varepsilon = 0$. Setting the derivative in $\varepsilon$ of this integral equal to zero we obtain $\int Bud\mu = 0$ for all $u \in C_+(\bar{G})$. This already implies that

$$
\int u(F^t x)d\mu(x) = \int ud\mu \quad \text{for all continuous } u, \tag{3.29}
$$

and so $\mu$ is $F^t$-invariant. If, on the other hand, the set $\mathcal{M}_G^F$ of $F^t$-invariant measures in $\bar{G}$ is empty then we cannot have $I_G^B(\mu) = 0$, and so the only possibility $I_G^B(\mu) = \infty$ is left. Then by (3.27), $\liminf_{i\to\infty} I_G^{\varepsilon_i}(\mu_i) = \infty$, and so (3.25) follows. We remark that $\mathcal{M}_G^F$ is empty if and only if there exist no

$F^t$-invariant sets in $\bar{G}$ since the support of any measure from $\mathcal{M}_G^F$ is an $F^t$-invariant set and for each $F^t$-invariant compact set $K$ one can find an $F^t$-invariant probability measure whose support is contained in $K$. Thus by the result from [Ki3] mentioned in Introduction together with (1.8) we derive that if $\mathcal{M}_G^F$ is not empty then

$$\liminf_{\varepsilon\to 0} \lambda_G^\varepsilon(V) > -\infty. \tag{3.30}$$

Now by (3.26) and (3.30) we have

$$I_G^\varepsilon(\mu_V^\varepsilon) = \int_G V d\mu_V^\varepsilon - \lambda_G^\varepsilon(V) \leqq \sup_{x\in G} |V(x)| - \liminf_{\varepsilon\to 0} \lambda_G^\varepsilon(V) + 1 \leqq \tilde{C} \tag{3.31}$$

for some $\tilde{C} < \infty$ and all $\varepsilon$ small enough. Hence (3.24) is satisfied, and so by the first part of the proof any weak limit of measures $\mu_V^\varepsilon$ as $\varepsilon \to 0$ belongs to $\mathcal{M}_G^F$. □

Remark 3.1. The same proof gives the lower semicontinuity of functionals $I_G^{\mathcal{L}}(\mu)$ in $\mathcal{L}$ with respect to the convergence of operators in the sense of the uniform convergence of their coefficients.

As a consequence of Theorems 2.1, 2.2 and Propositions 3.2 and 3.3 we obtain

Theorem 3.4. *Suppose that the maximal invariant set of the flow $F^t$ in $\bar{G}$ consists of a finite number of basic hyperbolic sets $K_i \subset G$, $i = 1, \ldots, \kappa$. Then for any $V \in C(\bar{G})$ all limit points as $\varepsilon \to 0$ of measures $\mu_V^\varepsilon$ are equilibrium states $\mu_V^0$ of the flow $F^t$ corresponding to the function $\varphi^u + V$ (with $\varphi^u$ extended continuously from $K_i$, $i = 1, \ldots, \kappa$ to the whole $\bar{G}$), i.e. $F^t$-invariant probability measures satisfying*

$$\begin{aligned} P_G(F, \varphi^u + V) &= \sup_{\nu\in\mathcal{M}_G^F} \left( h_\nu(F^1) + \int_G (\varphi^u + V) d\nu \right) \\ &= h_{\mu_V^0}(F^1) + \int_G (\varphi^u + V) d\mu_V^0. \end{aligned} \tag{3.32}$$

*All equilibrium states $\mu_V^0$ have the form*

$$\mu_V^0 = \sum_{1\leqq i\leqq\kappa} p_i \mu_{V,K_i}^0, \qquad p_i \geqq 0, \quad \sum_{1\leqq i\leqq\kappa} p_i = 1 \tag{3.33}$$

*where $\mu_{V,K_i}^0$ is an equilibrium state of the flow $F^t$ on $K_i$ corresponding to the function $\varphi^u + V$ and, if $p_j > 0$, then*

$$P_{K_j}(F, \varphi^u + V) = P_G(F, \varphi^u + V) = \max_{1 \leq i \leq \kappa} P_{K_i}(F, \varphi^u + V).$$

*If $V \equiv 0$ and there are attractors among $K_i$ then this maximum is zero and it is attained on attractors only. If this maximum is attained only for one $j$ and there exists a unique equilibrium state $\mu^0_{V,K_j}$ on $K_j$ corresponding to $\varphi^u + V$ then*

$$\mu^\varepsilon_V \xrightarrow{w} \mu^0_{V,K_j} \quad \text{as } \varepsilon \to 0. \tag{3.34}$$

*If $V$ is Hölder continuous then each $\mu^0_{V,K_i}$ is uniquely defined.*

Proof. It is known that the entropy $h_\nu(F^1)$ of the flow $F^t$ restricted to a hyperbolic set is upper semicontinuous (since $F^t$ is $h$-expansive, see [B], Example 1.6* and [DGS], Theorem 20.9). Set $J^{(0)}(\nu) = -h_\nu(F^1) - \int \varphi^u d\nu$ if $\nu$ is $F^t$-invariant and $J^{(0)}(\nu) = \infty$ otherwise. Remark that $J^{(0)}(\nu) \geqq 0$ since $-\int \varphi^u d\nu$ is the sum of positive Lyapunov exponents which by Ruelle's inequality is not less than the entropy. Clearly, $J^{(0)}(\nu)$ turns out to be a lower semicontinuous convex functional in $\nu$ since the entropy $h_\nu(F^1)$ is affine in $\nu$ (see [W]). By Theorems 2.1 and 2.2 together with the formula (2.7) we have

$$\lim_{\varepsilon \to 0} \lambda^\varepsilon_G(V) = P_G(F, \varphi^u + V). \tag{3.35}$$

Setting $J^{(i)}(\nu) = I^{\varepsilon_i}_G(\nu)$, $\lambda^{(i)}(V) = \lambda^{\varepsilon_i}_G(V)$ in Proposition 3.2 for $\varepsilon_i \to 0$ we derive from Propositions 3.2 and 3.3 that all limit points of measures $\mu^\varepsilon_V$ are equilibrium states of the flow $F^t$ corresponding to $\varphi^u + V$. The representation (3.33) follows from (2.7) which also implies that if $p_j > 0$ for some $j$ then $P_{K_j}(F, \varphi^u + V)$ must attain the maximum. From [BR] we know the uniqueness of equilibrium states corresponding to Hölder continuous functions for flows on basic hyperbolic sets and that $P_{K_i}(F, \varphi^u) = 0$ if and only if $K_i$ is an attractor. The Hölder continuity of $\varphi^u$ in the hyperbolic case is well known (see [BR]). □

Remark 3.2. To prove Theorem 3.4 one needs the upper semicontinuity of the entropy and some estimates of probabilities for diffusions $X^\varepsilon_t$ to stay in small tube neighborhoods of orbits of the flow $F^t$. Both things can be established in more general circumstances, for instance, in the case of uniformly partially hyperbolic sets (see [Y]) which enables one to extend Theorem 3.4 to this situation though one has to be careful since the uniqueness of equilibrium states may not be available in this case.

Remark 3.3. If we take $V = -\varphi^u$ then $\lambda^\varepsilon_G(V)$ will converge as $\varepsilon \to 0$ to the maximum of topological entropies of $F^t$ on $K_i$, $i = 1, \ldots, \kappa$. If $\kappa = 1$ then $\lambda_G^\varepsilon(V)$ will converge to the topological entropy of $F^t$ on $K_1$ and $\mu_V^\varepsilon$ will converge to the measure with maximal entropy (Margulis measure).

REMARK 3.4. In the above theorem we derive the behavior of $\mu_V^\varepsilon$ as $\varepsilon \to 0$ as a result of the convergence of $\lambda_G^\varepsilon(V)$ to the topological pressure. In another work we propose to study the behavior of $\mu_V^\varepsilon$ as $\varepsilon \to 0$ directly in certain cases when the convergence of $\lambda_G^\varepsilon(V)$ is not available.

THEOREM 3.5. *For any* $V \in C(\bar{G})$,

$$\limsup_{\varepsilon \to 0} \lambda_G^\varepsilon(V) \leqq \sup_{\mu \in \mathcal{M}_G^F} \int_{\bar{G}} V d\mu. \tag{3.36}$$

*If* $\mathcal{M}_G^F = \varnothing$ *then one should take* $-\infty$ *in the right hand side of* (3.36).

*If* $G = M$ *is compact then also*

$$\liminf_{\varepsilon \to 0} \lambda_M^\varepsilon(V) \geqq \inf_{\mu \in \mathcal{M}_M^F} \int_M V d\mu. \tag{3.37}$$

PROOF. By Proposition 3.3 all weak limits as $\varepsilon \to 0$ of measures $\mu_V^\varepsilon$ are $F^t$-invariant, and so

$$\begin{aligned}\limsup_{\varepsilon \to 0} \lambda_G^\varepsilon(V) &= \limsup_{\varepsilon \to 0} \left( \int_{\bar{G}} V d\mu_V^\varepsilon - I_G^\varepsilon(\mu_V^\varepsilon) \right) \\ &\leqq \limsup_{\varepsilon \to 0} \int_{\bar{G}} V d\mu_V^\varepsilon \\ &\leqq \sup_{\mu \in \mathcal{M}_G^F} \int_{\bar{G}} V d\mu \end{aligned}$$

since $I_G^\varepsilon(\eta) \geqq 0$ for all $\eta \in \mathcal{P}(\bar{G})$, proving (3.36). If $\mathcal{M}_G^F = \varnothing$ then by (3.25) we derive $\lambda_G^\varepsilon(V) \to -\infty$ as $\varepsilon \to 0$. If $G = M$ and $M$ is compact then by (1.5), (1.6), and Jensen's inequality,

$$\begin{aligned}\lambda_M^\varepsilon(V) &= \lim_{t \to \infty} \frac{1}{t} \ln E_x^\varepsilon \exp\left( \int_0^t V(X_s^\varepsilon) ds \right) \\ &\geqq \lim_{t \to \infty} \frac{1}{t} \int_0^t V(X_s^\varepsilon) ds \qquad (3.38) \\ &= \int_M V d\mu^\varepsilon \end{aligned}$$

where $\mu^\varepsilon$ is the unique invariant measure of the diffusion $X_s^\varepsilon$ on the compact manifold $M$ which, recall, equals $\mu_W^\varepsilon$ with $W \equiv 0$. Since all weak limits as $\varepsilon \to 0$ of measures $\mu^\varepsilon$ are $F^t$-invariant then passing to $\liminf_{\varepsilon \to 0}$ in (3.38) we obtain (3.37). □

THEOREM 3.6. *Suppose that $\mathcal{M}_G^F$ consists of one measure $\mu^0$ only (the uniquely ergodic case) and either $G = M$ is compact or $K = \operatorname{supp} \mu^0 \subset G$, $\bar{G} \neq M$ and $K$ is an attractor for the flow $F^t$, i.e. there exists an open set $U \supset K$, $\bar{U} \subset G$ such that $\bigcap_{t \geqq 0} F^t U = K$ and for any other open set $\tilde{U} \supset K$ there is $T = T(\tilde{U})$ such that $F^t U \subset \tilde{U}$ for all $t \geqq T$. Then for each $V \in C(\bar{G})$,*

$$\lim_{\varepsilon \to 0} \lambda_G^\varepsilon(V) = \int_{\bar{G}} V d\mu^0. \tag{3.39}$$

PROOF. If $G = M$ is compact then (3.39) follows immediately from (3.36) and (3.37). Suppose now that $K = \operatorname{supp} \mu^0 \subset G$ is an attractor. Remark that since $\mu^0$ is the only $F^t$-invariant measure with support in $\bar{G}$ then $\bar{G}$ does not contain $F^t$-invariant sets other than $K$. Since the upper bound (3.36) remains still valid we need to derive only the lower bound. By (1.5), (1.6), and the Markov property, if $t = Nn(\varepsilon)$, $x \in U_\delta(K) = \{z : \operatorname{dist}(z, K) < \delta\}$, and $\delta > 0$ is small enough, then

$$\begin{aligned} \lambda_G^\varepsilon(V) & \\ &\geqq \lim_{N \to \infty} \frac{1}{Nn(\varepsilon)} \ln Q_1^\varepsilon(Nn(\varepsilon), x, V, G) \\ &= \lim_{N \to \infty} \frac{1}{Nn(\varepsilon)} \ln E_x^\varepsilon \chi_{\tau_G^\varepsilon > n(\varepsilon)} \exp\left(\int_0^{n(\varepsilon)} V(X_s^\varepsilon) ds\right) E^\varepsilon_{X^\varepsilon_{n(\varepsilon)}} \chi_{\tau_G^\varepsilon > n(\varepsilon)} \exp\left(\int_0^{n(\varepsilon)} V(X_s^\varepsilon) ds\right) \\ &\qquad \times \cdots \times E^\varepsilon_{X^\varepsilon_{(N-1)n(\varepsilon)}} \chi_{\tau_G^\varepsilon > n(\varepsilon)} \exp\left(\int_0^{n(\varepsilon)} V(X_s^\varepsilon) ds\right) \\ &\geqq \frac{1}{n(\varepsilon)} \ln \inf_{z \in U_\delta(K)} E_z^\varepsilon \chi_{\tau_G^\varepsilon > n(\varepsilon)} \chi_{X^\varepsilon_{n(\varepsilon)} \in U_\delta(K)} \exp\left(\int_0^{n(\varepsilon)} V(X_s^\varepsilon) ds\right) \\ &\geqq \frac{1}{n(\varepsilon)} \ln \inf_{z \in U_\delta(K)} E_z^\varepsilon \chi_{\tau^\varepsilon_{U_{\gamma_1(\delta)}(K)} > n(\varepsilon)} \chi_{X^\varepsilon_{n(\varepsilon)} \in U_\delta(K)} \exp\left(\int_0^{n(\varepsilon)} V(X_s^\varepsilon) ds\right) \end{aligned} \tag{3.40}$$

where $U_{\gamma_1(\delta)}(K) \subset G$, $\gamma_1(\delta) \downarrow 0$ as $\delta \downarrow 0$, $n(\varepsilon)$ is the integral part of $\varepsilon^{-1/4}$, and additional properties of $\gamma_1(\delta)$ will be specified below. Applying arguments of Proposition 4.2 of Chapter I from [Ki4] (which hold true whenever $K$ is an attractor and $F^t$ is a continuous flow) we can choose $\gamma_1(\delta)$ so that for $\varepsilon < \varepsilon(\delta)$

any $\varepsilon^{1/4}$-pseudo-orbit of $F^t$ starting in $U_\delta(K)$ stays forever in $U_{(\gamma_1(\delta))^2}(K)$ and any $\varepsilon^{1/4}$-pseudo-orbit of length at least $n(\varepsilon)$ starting in $U_\delta(K)$ ends in $U_\delta(K)$, as well (in fact, it will end arbitrary close to $K$ depending on how small $\varepsilon$ is). By the inequalities (4.4) and (4.5) of the next section we conclude that except for the probability of order $\exp(-\beta_1\varepsilon^{-1/2})$ for some $\beta_1>0$ paths of the process $X_t^\varepsilon$ considered at integer moments $t=0,1,2,\ldots,n(\varepsilon)$ are $\varepsilon^{1/4}$-pseudo-orbits. Besides, from the estimate (5.8) of [Ki2] and from the $F^t$-invariance of $K$ it follows that if $\delta$ is small enough and $X_k^\varepsilon$, $k=0,\ldots,n(\varepsilon)$ stays in $U_{(\gamma_1(\delta))^2}(K)$ then $X_t^\varepsilon\in U_{\gamma_1(\delta)}(K)$ for all $t\in[0,n(\varepsilon)]$ except for the probability of order $n(\varepsilon)\exp(-\beta_2(\gamma_1(\delta))^2\varepsilon^{-1})$. From these we derive that for each $z\in U_\delta(K)$ and any $\varepsilon$ small enough $\Theta\supset\Xi$ where $\Theta$ and $\Xi$ are events defined by

$$\Theta=\{\tau^\varepsilon_{U_{\gamma_1(\delta)}(K)}>n(\varepsilon)\text{ and }X^\varepsilon_{n(\varepsilon)}\in U_\delta(K)\},$$

$$\Xi=\left\{\max_{0\leqq k\leqq n(\varepsilon)}\ \sup_{0\leqq s\leqq 1}\ \mathrm{dist}(F^sX_k^\varepsilon,X_{k+s}^\varepsilon)\leqq C\varepsilon^{1/2}\right\},$$

$X_0^\varepsilon=z\in U_\delta(K)$, and moreover by (5.8) of [Ki2],

$$P_z^\varepsilon\{\Theta\}\geqq P_z^\varepsilon\{\Xi\}\geqq 1-2n(\varepsilon)\exp(-\beta_1\varepsilon^{-1/2}). \tag{3.41}$$

Here one can take $C=2\sup_{-1\leqq s\leqq 1}\|DF^s\|$ and $DF^s$ denotes the differential of $F^s$.

If the event $\Xi$ occurs then we can split each path of $X_t^\varepsilon$, $0\leqq t\leqq n(\varepsilon)$ into pieces $iS(\delta)\leqq t<(i+1)S(\delta)$, $i=0,\ldots,k(\varepsilon,\delta)-1$, $S(\delta)=n(\varepsilon)/k(\varepsilon,\delta)$ so that $S(\delta)\uparrow\infty$ as $\delta\downarrow 0$ and for each $\omega\in\Xi$ there are points $z_i(\omega)\in K$, $i=0,\ldots,k(\varepsilon,\delta)-1$ satisfying

$$\sup_{0\leqq u\leqq S(\delta)}\ \mathrm{dist}(F^uz_i(\omega),X^\varepsilon_{iS(\delta)+u}(\omega))\leqq\sqrt{\gamma_1(\delta)}. \tag{3.42}$$

Since $V$ is a continuous function then there is $\gamma_2(\delta)\downarrow 0$ as $\delta\downarrow 0$ such that for each $\omega$ satisfying (3.42) one has

$$\left|\int_0^{n(\varepsilon)}V(X_s^\varepsilon(\omega))ds-\sum_{i=0}^{k(\varepsilon,\delta)-1}\int_0^{S(\delta)}V(F^uz_i(\omega))du\right|\leqq n(\varepsilon)\gamma_2(\delta). \tag{3.43}$$

Since $F^t$ is a uniquely ergodic flow then by the continuous time version of Theorem 6.19 from [W] we derive that there exists $\gamma_3(\delta)\downarrow 0$ as $\delta\downarrow 0$ such that uniformly in $z\in K$,

$$\left|\frac{1}{S(\delta)}\int_0^{S(\delta)}V(F^uz)du-\int_K Vd\mu^0\right|\leqq\gamma_3(\delta). \tag{3.44}$$

Finally, (3.40)–(3.44) together with Jensen's inequality yield for $\varepsilon$ sufficiently small that

$$
\begin{aligned}
\lambda_G^\varepsilon(V) &\geqq \frac{1}{n(\varepsilon)} \inf_{z\in U_\delta(K)} \ln E_z^\varepsilon \chi_\Xi \exp\left(\int_0^{n(\varepsilon)} V(X_s^\varepsilon)ds\right) \\
&\geqq \int_K V d\mu^0 - \gamma_2(\delta) - \gamma_3(\delta) + \frac{1}{n(\varepsilon)} \ln(1-\exp(-\varepsilon^{-1/4})).
\end{aligned} \tag{3.45}
$$

Letting in (3.45) first $\varepsilon \to 0$ and then $\delta \to 0$ we obtain $\liminf_{\varepsilon\to 0} \lambda_G^\varepsilon(V) \geqq \int_K V d\mu^0$ which together with (3.36) proves (3.39) since $K = \operatorname{supp} \mu^0$. □

REMARK 3.5. For the hyperbolic case we obtained that $\lambda_G^\varepsilon(V)$ converges to the topological pressure corresponding to the function $V + \varphi^u$ and in more general cases I believe that $\varphi^u$ can be replaced by the minus sum of positive Lyapunov exponents. If we have a uniquely ergodic flow with all exponents being zero then Theorem 3.5 supports this conjecture. All known smooth examples of uniquely ergodic flows have zero exponents but nobody knows whether it is always so. Thus the above conjecture is connected with existence of smooth uniquely ergodic flows with nonzero exponents since for such flows the above conjecture would agree with Theorem 3.5 only if the unique invariant measure satisfies Pesin's formula, and so the entropy and the sum of positive exponents cancel each other in the formula for the topological pressure.

One can see that the behavior of the functional $I_G^\varepsilon(\mu)$ for small $\varepsilon$ is natural also from the point of view of large derivations. By [DV2] the functional $I_G^\varepsilon(\mu)$ describes the large derivations behavior near $\mu$ as $t \to \infty$ of the occupational measure

$$
\zeta^\varepsilon(t,\omega,A) = \frac{1}{t}\int_0^A \chi_A(X^\varepsilon_{\min(s,\tau_G^\varepsilon)}(\omega))ds, \qquad A \subset G.
$$

Namely, if $Q_{x,t}^\varepsilon(B) = P_x^\varepsilon\{\zeta^\varepsilon(t,\omega,\cdot)\in B\}$, $B \subset \mathscr{P}(G)$, then for any closed set $\mathscr{C} \subset \mathscr{P}(G)$,

$$
\limsup_{t\to\infty} \frac{1}{t} \ln Q_{x,t}^\varepsilon(\mathscr{C}) \leqq -\inf_{\mu\in G} I_G^\varepsilon(\mu)
$$

and for any open set $\mathscr{O} \subset \mathscr{P}(G)$,

$$
\liminf_{t\to\infty} \frac{1}{t} \ln Q_{x,t}^\varepsilon(\mathscr{O}) \geqq -\inf_{\mu\in\mathscr{O}} I_G^\varepsilon(\mu).
$$

On the other hand, results of [OP] and [Y] tell us that if $K$ is a hyperbolic attractor then

$$I_K(\eta) = \begin{cases} -h_\eta(F^1) - \int_K \varphi^u d\eta & \text{if } \eta \in \mathcal{M}_K^F \\ \infty & \text{otherwise} \end{cases}$$

is the rate function for occupational measures

$$\zeta^0(t, \omega, A) = \frac{1}{t}\int_0^t \chi_A(F^s x)ds,$$

i.e., again, for any closed set $\mathscr{C} \subset \mathscr{P}(U)$,

$$\lim_{t\to\infty} \frac{1}{t} \ln m\{x \in U : \zeta^0(t,. x, \cdot) \in C\} \leqq -\inf_{\eta \in C} I_K(\eta)$$

and for any open set $\mathscr{O} \subset \mathscr{P}(U)$,

$$\lim_{t\to\infty} \frac{1}{t} \ln m\{x \in U : \zeta^0(t, x, \cdot) \in \mathscr{O}\} \geqq -\inf_{\eta \in \mathscr{O}} I_K(\eta)$$

where $m$ is the Lebesgue measure and $U$ is a small neighborhood of $K$. So the transformation of $-I_G^\varepsilon(\mu)$ into $I_K(\mu)$ as $\varepsilon \to 0$ seems natural. This is on the first sight the meaning of (2.6) considering $\lambda_G^\varepsilon(V)$ in the form (1.16) and $P_K(F, \varphi^u + V)$ in the form (1.19). But the things are not so simple since usually $I_G^\varepsilon(\mu) = \infty$ if $\mu$ is not absolutely continuous with respect to $m$ (see [DV2]) and $I_K(\mu) = \infty$ if $\mu$ is not $F^t$-invariant. Since most (if not all) $F^t$-invariant measures are not absolutely continuous then as a rule we can not obtain $I_K(\mu)$ as a limit of $I_G^\varepsilon(\mu)$ as $\varepsilon \to 0$ for fixed $\mu$. The correct approach to this limit is described by Theorem 2.2 and Proposition 3.3.

## 4. Proofs of Theorems 2.1 and 2.2 and Proposition 2.4

We will start with Theorem 2.1. Since $\tau_{U_i \cap G}^\varepsilon \leqq \tau_G^\varepsilon$ (if $G = M$ then $\tau_G^\varepsilon = \infty$ and so this is also true) we conclude from (1.5) and (1.6) that $\lambda_G^\varepsilon(V) \geqq \lambda_{U_i}^\varepsilon(V)$ for all $i = 1, \ldots, \kappa$, and so

$$\liminf_{\varepsilon\to 0} \lambda_G^\varepsilon(V) \geqq \max_{1\leqq i\leqq\kappa} \lambda_{K_i}(V). \tag{4.1}$$

Therefore it remains only to estimate $\lambda_G^\varepsilon(V)$ from above. By (1.4) and the definition (1.6) of $Q_1^\varepsilon(t, x, V, G)$,

$$\lambda_G^\varepsilon(V) = \inf_{t>0} \frac{1}{t} \ln \sup_{x \in G} Q_1^\varepsilon(t, x, V, G). \tag{4.2}$$

Denote

$$Q_2^\varepsilon(n, \delta, x, V, G) = E_x^\varepsilon \left( \prod_{k=0}^{n-1} \chi_{X_{k+1}^\varepsilon \in U_\delta(F^1 X_k^\varepsilon)}) \chi_{\tau_G^\varepsilon > n} \exp\left( \int_0^n V(X_s^\varepsilon) ds \right) \tag{4.3}$$

where $U_\delta(y) = \{z : \operatorname{dist}(z, y) < \delta\}$ and $\Pi$ denotes the product. Then by the Markov property for any integer $n > 0$,

$$\begin{aligned} &| Q_1^\varepsilon(n, x, V, G) - Q_2^\varepsilon(n, \delta, x, V, G) | \\ &\leqq n e^{C_0 n} \sup_{z \in G} P_z^\varepsilon\{\operatorname{dist}(F^1 z, X_1^\varepsilon) \geqq \delta\} \end{aligned} \tag{4.4}$$

where $C_0$ was defined in (1.8). By estimates of [A] it follows that

$$\mathrm{P}_z^\varepsilon\{\operatorname{dist}(F^1 z, X_1^\varepsilon) \geqq \delta\} \leqq C_2 \varepsilon^{-\nu} e^{-\beta_2 \delta^2 \varepsilon^{-1}} \tag{4.5}$$

for any $z \in G$ and some constants $C_2$, $\beta_2 > 0$. Now if (2.3) holds for $t(\varepsilon, \delta) \leqq \varepsilon^{-2(1-\beta_0)}$ we choose $n = n(\varepsilon) = [\varepsilon^{-2+\beta_0}]$ where $[\cdot]$ denotes the integral part. By (4.2),

$$\lambda_G^\varepsilon(V) \leqq \frac{1}{n} \ln \sup_{x \in G} Q_1^\varepsilon(n, x, V, G). \tag{4.6}$$

In the same way as in [EK] one concludes from (4.3)–(4.5) that in order to estimate the right hand side of (4.6) with $n = n(\varepsilon)$ it suffices to take into account only those paths of the diffusion $X_t^\varepsilon$ which are $\delta$-pseudo-orbits, i.e. such that

$$\operatorname{dist}(F^1 X_k^\varepsilon, X_{k+1}^\varepsilon) \leqq \delta \quad \text{for all } k = 0, 1, \ldots, n(\varepsilon) - 1. \tag{4.7}$$

From this point the proof of Theorem 2.1 proceeds exactly in the same way as the proof of Theorem 2.1 from [EK] employing the Markov property of the process $X_t^\varepsilon$ (or the semigroup property of the operators $T_{V,G}^\varepsilon$) in place of the Chapman–Kolmogorov formula in [EK].

Next we will discuss the proof of Theorem 2.2. Denote

$$Q_3^\varepsilon(t, \delta, x, y, V, U) = E_x^\varepsilon \chi_{\tau_U^\varepsilon > t} \chi_{\sup_{0 \le s \le t} \operatorname{dist}(X_s^\varepsilon, F^s y) \le \delta} \exp\left( \int_0^t V(X_s^\varepsilon) ds \right). \tag{4.8}$$

Then

$$
\begin{aligned}
e^{-t\gamma_U(V,\delta)} &\leqq Q_3^\varepsilon(t,\delta,x,y,V,U)\left(Q_3^\varepsilon(t,\delta,x,y,0,U)\exp\left(\int_0^t V(F^s y)ds\right)\right)^{-1} \\
&\leqq e^{t\gamma_U(V,\delta)}
\end{aligned}
\tag{4.9}
$$

where

$$
\gamma_U(\psi,\delta)=\exp\{|\psi(z_1)-\psi(z_2)|: z_1, z_2\in U, \operatorname{dist}(z_1,z_2)\leqq\delta\}
\tag{4.10}
$$

for some $C_3$, $\beta_2>0$ since we assume that $V$ is continuous. By Theorem 2.1 of [Ki2] there exist positive constants $\varepsilon_0$, $\delta_0$, $\alpha_1$ and $\alpha_2$ so that

$$
\begin{aligned}
(1-\varepsilon^{\alpha_1})^t\exp(-\alpha_2/\varepsilon^2) &\leqq Q_3^\varepsilon(t,\delta,x,y,0,U)\exp\left(-\int_0^t \varphi^u(F^s y)ds\right) \\
&\leqq (1+\varepsilon^{\alpha_1})^t
\end{aligned}
\tag{4.11}
$$

for any $x\in M$, $y\in K$ whenever $\operatorname{dist}(x,y)\leqq\delta/2$, $0\leqq\varepsilon\leqq\varepsilon_0$, $\varepsilon^{2/3}\leqq\delta\leqq\delta_0$, and $t\geqq\varepsilon^{-1/10}$ (the right hand side of (4.11) holds true without the assumption $\operatorname{dist}(x,y)\leqq\delta/2$ and $\varepsilon^{-1/10}$ can be replaced by $\varepsilon^{-\alpha}$ with $\alpha>0$ arbitrarily small). If $\operatorname{dist}(x,y)\leqq\varepsilon$ then one can put $\alpha_2=0$.

Remark 4.1. The reader willing to go through the details of the proof of Theorem 2.1 in [Ki2] should be warned about a mistake in the definition of the operator $\widetilde{DS^t}$ in the formula (4.12) there. The restriction of $\widetilde{DS^t}$ on $F^s$ should be taken as an arbitrary linear operator $\widetilde{DS^t}: F_x^s\to F_{S^t x}^s$ preserving inner products there.

In what follows we will need the notion of a $(\delta,T)$-separated set which is a subset $S$ of $K$ such that $x,y\in S$, $x\neq y$ imply $\operatorname{dist}(F^t x,F^t y)>\delta$ for some $t\in[0,T]$. It is clear that if $S$ is a $(\delta,T)$-separated set and $y,z\in S$, $y\neq z$ then $U_{\delta/2}(y,T)$ and $U_{\delta/2}(z,T)$ are disjoint where

$$
U_{\delta/2}(v,T)=\{w:\operatorname{dist}(F^t v,F^t w)\leqq\delta/2 \text{ for all } t\in[0,T]\}.
$$

Thus $S$ is a finite set and the number of points of $S$ does not exceed some constant $l(\delta,T)$ depending just on $\delta$ and $T$. Thus there exists a maximal $(\delta,T)$-separated set $S(\delta,T)$ (not necessarily unique). It is clear that for any $y,z\in S(\delta,T)$, $y\neq z$ the events $\{\operatorname{dist}(X_t^\varepsilon,F^t y)\leqq\delta/2$ for all $t\in[0,T]\}$ and $\{\operatorname{dist}(X_t^\varepsilon,F^t z)\leqq\delta/2$ for all $t\in[0,T]\}$ are inconsistent. Thus if a neighborhood $U$ of $K$ appearing in Theorem 2.2 satisfies $U\supset U_{\delta/2}(K)=\{z:\operatorname{dist}(z,K)\leqq\delta/2\}$ then

$$Q_1^\varepsilon(T,x,V,U) \geqq \sum_{y\in S(\delta,T)} Q_3^\varepsilon(T,\delta/2,x,y,V,U) \tag{4.12}$$

for any $x \in U$.

Denote by $p_U^\varepsilon(t,x,y)$ the transition density with respect to the Lebesgue measure $m$ on $M$ of the part of the process $X_t^\varepsilon$ until the first exit from $U$, i.e. in other words, the fundamental solution of the problem $\partial u/\partial t = L^\varepsilon u$ with zero data on the boundary of $U$ (see [F2], Chapter 6). Let $U_1 \supset U_{\delta/4}(K)$ be an open set such that $\bar U_1 \subset U$. Then by the strong maximum principle and the continuity of $p_U^\varepsilon(t,x,y)$ for any $x \in U$ one has

$$\inf_{z\in U_1} p_U^\varepsilon(1,x,z) = p_{U_1}^\varepsilon(x) > 0. \tag{4.13}$$

Now from (1.6) and (4.8)–(4.13) it follows that

$$\begin{aligned}
Q_1^\varepsilon(T,x,V,U) &\geqq e^{-C_0} Q_1^\varepsilon(T+1,x,V,U)\\
&\geqq e^{-2C_0}\int_{U_1} p_U^\varepsilon(1,x,z) Q_1^\varepsilon(T,z,V,U)dm(z)\\
&\geqq e^{-2C_0} p_{U_1}^\varepsilon(x)\int_{U_1} Q_1^\varepsilon(T,z,V,U)dm(z)\\
&\geqq e^{-2C_0} p_{U_1}^\varepsilon(x)\int_{U_1}\sum_{y\in S(\delta,T)} Q_3^\varepsilon(T,\delta/2,z,y,V,U)dm(z) \qquad (4.14)\\
&\geqq e^{-2C_0} p_{U_1}^\varepsilon(x)\sum_{y\in S(\delta,T)}\int_{U_{\delta/4}(y)} Q_3^\varepsilon(T,\delta/2,z,y,V,U)dm(z)\\
&\geqq e^{-2C_0} p_{U_1}^\varepsilon(x)(1-\varepsilon^{\alpha_1})^T \exp(-\alpha_2/\varepsilon^2)e^{-T\gamma_U(V,\delta)}\min_{y\in S(\delta,T)} m(U_{\delta/2}(y))\\
&\quad\times \sum_{y\in S(\delta,T)} \exp\left(\int_0^T (\varphi^u(F^s y) + V(F^s y))ds\right).
\end{aligned}$$

Thus by (1.5),

$$\begin{aligned}
\lambda_U^\varepsilon(V) \geqq \limsup_{T\to\infty} \frac{1}{T}\ln \sum_{y\in S(\delta,T)} \exp\left(\int_0^T (\varphi^u(F^s y) + V(F^s y))ds\right) - \varepsilon^{\alpha_1}\\
- \gamma_U(V,\delta). \qquad (4.15)
\end{aligned}$$

Next, we will recall another definition of the topological pressure $P_K(F,\psi)$ which is equivalent to (2.5) (see [W], Chapter 9). Define

$$Z_K(F,\psi,\delta,T) = \sup\left\{\sum_{y\in S}\exp\left(\int_0^T \psi(F^t y)dt : S \text{ is } (\delta,T)\text{-separated}\right\}\right. \tag{4.16}$$

and

$$P_K(F, \psi, \delta) = \limsup_{T \to \infty} \frac{1}{T} \ln Z_K(F, \psi, \delta, T). \tag{4.17}$$

Then

$$P_K(F, \psi) = \lim_{\delta \to 0} P_K(F, \psi, \delta) \tag{4.18}$$

where $P_K(F, \psi)$ is the same as in (2.5). Let again $S(\delta, T)$ be a maximal $(\delta, T)$-separated set in $K$ and let $S$ be any $(2\delta, T)$-separated set in $K$. Then, clearly, for each point $z \in S$ there is a point $y \in S(\delta, T)$ such that

$$\sup_{0 \leqq t \leqq T} \operatorname{dist}(F^t y, F^t z) \leqq \delta.$$

This point $y$ is not necessarily unique but to different points $z$ from $S$ must correspond different points $y$ from $S(\delta, T)$ since $S$ is $(2\delta, T)$-separated. One concludes from above that

$$\sum_{y \in S(\delta, T)} \exp\left(\int_0^T \psi(F^t y)dt\right) \geqq Z_K(F, \psi, 2\delta, T)e^{-T\gamma_K(\psi, \delta)}, \tag{4.19}$$

where $\gamma_K(\psi, \delta)$ was defined by (4.10). This together with (4.16) and (4.17) yields

$$\begin{aligned} P_K(F, \psi, \delta) &\geqq \limsup_{T \to \infty} \left( \sum_{y \in S(\delta, T)} \exp\left(\int_0^T \psi(F^t y)dt\right)\right) \\ &\geqq P_K(F, \psi, 2\delta) - \gamma_K(\psi, \delta). \end{aligned} \tag{4.20}$$

So for a continuous $\psi$ one gets the topological pressure in (4.18) without taking the supremum in (4.16), just by considering the sum there over any maximal $(\delta, T)$-separated set.

In our circumstances (4.15) and (4.20) imply

$$\lambda_U^\varepsilon(V) \geqq P_K(F, \varphi^u + V, 2\delta) - \gamma_K(\varphi^u + V, \delta) - \gamma_U(V, \delta) - \varepsilon^{\alpha_1}. \tag{4.21}$$

First letting $\varepsilon \to 0$ and then $\delta \to 0$ we obtain

$$\liminf_{\varepsilon \to 0} \lambda_U^\varepsilon(V) \geqq P_K(F, \varphi^u + V); \tag{4.22}$$

since $\varphi^u$ and $V$ are continuous then both $\gamma_K(\varphi^u + V, \delta)$ and $\gamma_U(V, \delta)$ tend to zero as $\delta \to 0$.

Next, we will obtain an upper bound for $Q_1^\varepsilon(T, x, V, U)$. Employing arguments from the proof of Theorem 2.1 with $G = U$, $\kappa = 1$, and $K_1 = K$ we see that we can substitute $U$ by any arbitrarily small neighborhood of $K$, i.e. it suffices to obtain estimates for $Q_1^\varepsilon(T, x, V, \tilde{U})$ where $\tilde{U} \supset K$, $\tilde{U} \subset U_{\delta_1}(K)$ with $\delta_1 > 0$ small enough is an open set with a piecewise smooth boundary. In view of (4.2) it suffices to obtain an appropriate upper bound only for some $T$ which will be chosen as $T = n = n(\varepsilon) = [\varepsilon^{-1/10}] + 1$. Take $\delta = \delta(\varepsilon) = \varepsilon^{1/2}$, then we conclude from (4.4) and (4.5) that it suffices to estimate $Q_2^\varepsilon(n(\varepsilon), \delta(\varepsilon), x, V, \tilde{U})$ in place of $Q_1^\varepsilon(n(\varepsilon), x, V, \tilde{U})$, i.e. one has to take into account only paths of the process $X_t^\varepsilon$ which are $\delta(\varepsilon)$-pseudo-orbits staying in $\tilde{U}$. Since we are in a small neighborhood of a hyperbolic set $K$ the shadowing lemma (Lemma 3.2 of [Ki2]) says that there exists a constant $C_3 > 0$ such that for any $\delta$-pseudo-orbit $\omega = (x_0, x_1, \ldots, x_{n(\varepsilon)})$ with $x_i \in \tilde{U} \subset U_{\delta_1}(K)$, $i = 0, \ldots, n(\varepsilon)$ one can find a point $y_\omega$ such that

$$\operatorname{dist}(F^k y_\omega, x_k) \leqq C_3 n(\varepsilon)\delta(\varepsilon) \quad \text{for all } k = 0, \ldots, n(\varepsilon). \tag{4.23}$$

By Lemma 3.1 of [Ki2] there exists a constant $C_4 > 0$ independent of $\omega = (x_0, \ldots, x_{n(\varepsilon)})$ and a point $z_\omega \in K$ such that

$$\operatorname{dist}(F^k z_\omega, x_k) \leqq C_4 \delta_1 \quad \text{for all } k = 0, \ldots, n(\varepsilon). \tag{4.24}$$

Put $\delta_2 = C_4 \delta_1$ and consider a maximal $(\delta_2, n(\varepsilon))$-separated set $S(\delta_2, n(\varepsilon))$. Then

$$\begin{aligned} &Q_2^\varepsilon(n(\varepsilon), \delta(\varepsilon), x, V, \tilde{U}) \\ (4.25)\quad &\leqq \sum_{y: y \in S(\delta_2, n(\varepsilon)), y \in U_{2\delta_2}(x)} E_x^\varepsilon \chi_{\tau_{\tilde{U}}^\varepsilon > n(\varepsilon)} \chi_{\max_{1 \leq j \leq n(\varepsilon)} \operatorname{dist}(X_j^\varepsilon, F^j y) \geqq 2\delta_2} \exp\left(\int_0^{n(\varepsilon)} V(X_s^\varepsilon)\,ds\right) \\ &= \sum_{y: y \in S(\delta_2, n(\varepsilon)), y \in U_{2\delta_2}(x)} Q_3^\varepsilon(n(\varepsilon), 4C_5\delta_2, x, y, V, \tilde{U}) + \mathfrak{R}^\varepsilon, \end{aligned}$$

where the last inequality is just the definition of $\mathfrak{R}^\varepsilon$ and

$$C_5 = \sup_{-1 \leqq t \leqq 1} \| DF^t \| \geqq 1. \tag{4.26}$$

Employing the Markov property we obtain

$$\begin{aligned} \mathfrak{R}^\varepsilon &\leqq l(\delta_2, n(\varepsilon)) r(\varepsilon) e^{C_0 n(\varepsilon)} \\ &\times \sup_{y \in K, z \in U_{2\delta_2}(y)} P_z^\varepsilon \left\{ \sup_{0 \leqq t \leqq 1} \operatorname{dist}(X_t^\varepsilon, F^t y) > 4C_5\delta_2 \right\} \end{aligned} \tag{4.27}$$

where, recall, $l(\delta_2, n(\varepsilon))$ is the number of points in $S(\delta_2, n(\varepsilon))$ which satisfies

$$l(\delta_2, n(\varepsilon)) \leqq (C_6\delta_1^{-\gamma})^{n(\varepsilon)} \tag{4.28}$$

for some $C_6 > 1$. In the same way as in [Ki2] we use Theorem 1.2 from [VF] to derive that the probability in the right hand side of (4.27) does not exceed $\exp(-\alpha_3/\varepsilon^2)$ for some $\alpha_3 > 0$ provided $\varepsilon$ is small enough and, say, $\delta_1 \geqq \varepsilon^{1/2}$. Then by (4.27) and (4.28) for small $\varepsilon$,

$$\mathfrak{R}^\varepsilon \leqq \exp\left(-\frac{\alpha_3}{2\varepsilon^2}\right). \tag{4.29}$$

Finally using the right hand sides of (4.9) and (4.11) together with (4.4), (4.5), (4.25), and (4.29) we obtain for $\varepsilon$ and $\delta_1$ small enough, $\delta_1 \geqq \varepsilon^{1/2}$ that

$$\begin{aligned} Q_1^\varepsilon(n(\varepsilon), c, V, \tilde{U}) &\leqq Q_2^\varepsilon(n(\varepsilon), \delta(\varepsilon), x, V, \tilde{U}) + e^{-\varepsilon^{-1/2}} \\ &\leqq 2\exp(-\varepsilon^{1/2}) + (1+\varepsilon^{\alpha_1})^{n(\varepsilon)}\exp(n(\varepsilon)\gamma_u(V, 4C_4C_5\delta_1)) \\ &\quad \times \sum_{y:y\in S(C_4\delta_1, n(\varepsilon))} \exp\left(\int_0^{n(\varepsilon)} (\varphi^u(F^s y) + V(F^s y))ds\right). \end{aligned} \tag{4.30}$$

From (4.6) which holds also for $G = \tilde{U}$ and from (4.16)–(4.17) we derive

$$\limsup_{\varepsilon\to 0} \lambda_{\tilde{U}}^\varepsilon(V) \leqq P_K(F, \varphi^u + V, C_4\delta_1) + \gamma_U(V, 4C_4C_5\delta_1). \tag{4.31}$$

Letting $\delta_1 \to 0$ one obtains

$$\limsup_{\varepsilon\to 0} \lambda_{\tilde{U}}^\varepsilon(V) \leqq P_K(F, \varphi^u + V). \tag{4.32}$$

By arguments of the proof of Theorem 2.1 given in the proof of Theorem 2.1 in [EK] it follows that the left hand side of (4.32) remains the same if $\tilde{U}$ is replaced by $U$. This together with (4.22) gives (2.7) and completes the proof of Theorem 2.2. □

Next, we will etablish Proposition 2.4. If $w_g^\varepsilon$ solves (1.11) and (1.14) holds true, then by the comparison theorem (see [F1], Chapter 2) $|w_g^\varepsilon| \leqq |v_g^\varepsilon|$ where $v_g^\varepsilon$ solves the equation

$$\begin{cases} \dfrac{\partial v_g^\varepsilon(t,x)}{\partial t} = L^\varepsilon v_g^\varepsilon(t,x) + C_0 v_g^\varepsilon(t,x), \\ v_g^\varepsilon\big|_{t=0} = g, \quad v_g^\varepsilon\big|_{x\in\partial G} = 0. \end{cases} \tag{4.33}$$

If $G = M$ and $M$ is compact then the boundary condition in (4.33) should be disregarded. Since $C_0 < -\lambda_G^\varepsilon(0)$, $\lambda_G^\varepsilon(0) \leqq 0$ then, for large $t$,

$$\| w_g^\varepsilon(t,x) \| \leqq \| v_g^\varepsilon(t,x) \| \leqq e^{-a_\varepsilon t} \| g \| \xrightarrow[t\to\infty]{} 0 \tag{4.34}$$

where $a_\varepsilon = -\frac{1}{2}(\lambda_G^\varepsilon(0) + C_0) > 0$. Then solution of (1.11) satisfies the following integral equation:

$$w_g^\varepsilon(t,x) = E_x^\varepsilon \chi_{\tau_G^\varepsilon > t} g(X_t^\varepsilon) \exp\left( \int_0^t V(X_s^\varepsilon, w_g^\varepsilon(s, X_s^\varepsilon)) ds \right) \tag{4.35}$$

where $V(x, u)$ was defined by (1.12). Using uniform convergence in (1.13) and (2.13) we derive (1.15) and moreover if $g \geqq 0$ then

$$\lim_{t\to\infty} \frac{1}{t} \ln w_g^\varepsilon(t,x) = \lim_{t\to\infty} \frac{1}{t} \ln u_g^\varepsilon(t,x) \tag{4.36}$$

where $u_g^\varepsilon(t,x)$ solves the equation (1.3) with $V(x) = V_0(x) = \lim_{u\to 0} V(x,u)$. In order to obtain (2.14) it suffices therefore to prove the following result.

LEMMA 4.1. *Let $u_g^\varepsilon(t,x)$ be a solution* (1.3) *with $g \geqq 0$, $g \not\equiv 0$. Then*

$$\lim_{t\to\infty} \frac{1}{t} \ln u_g^\varepsilon(t,x) = \lambda_G^\varepsilon(V) \tag{4.37}$$

*for any $x \in G$.*

PROOF. Let $\tilde{V} = V - C_0$ with $C_0$ given by (1.8). Now if $\tilde{u}_g^\varepsilon$ is the solution of (1.3) with $\tilde{V}$ in place of $V$ then $\tilde{u}_g^\varepsilon(t,x) = e^{-C_0 t} u_g^\varepsilon(t,x)$ and $\lambda_G^\varepsilon(\tilde{V}) = \lambda_G^\varepsilon(V) - C_0$. Hence it suffices to prove (4.37) with $\tilde{V}$ in place of $V$, i.e. we can assume from the beginning that $V \leqq 0$. In this case we can apply the strong maximum-principle for parabolic equations (see [F1], Chapter 2) which yields for $t \geqq t_0 > 0$ that

$$r_{g,t_0}^{-1} Q_1^\varepsilon(t,x,V,G) \leqq u_g^\varepsilon(t,x) \leqq r_{g,t_0} Q_1^\varepsilon(t,x,V,G) \tag{4.38}$$

where $r_{g,t_0} < \infty$ is a positive constant and $Q_1^\varepsilon$ is given by (1.6). Indeed, both $Q_1^\varepsilon$ and $u_g^\varepsilon$ are solutions of the parabolic equation (1.3) with zero data on $\partial G$, they are smooth and positive in $G$, and by the strong maximum principle their outward normal derivatives are strictly negative along $\partial G$. It follows that both $Q_1^\varepsilon(t,x,V,G)(u_g^\varepsilon(t,x))^{-1}$ and $u_g^\varepsilon(t,x)(Q_1^\varepsilon(t,x,V,G))^{-1}$ are bounded in $x$ for any fixed $t > 0$ and so by the comparison theorem these ratios are bounded uniformly in $(t, x)$ provided $t \geqq t_0 > 0$, proving (4.38). Now (4.38) together with (1.5) gives (4.37) proving Lemma 4.1. The proof of (2.17) under the conditions (2.15), (2.16), and (2.18) is the same as the above proof of (2.14). □

REMARK 4.2. If some solution $w_g^\varepsilon$ of (1.11) satisfies $w_g^\varepsilon(t, x) \to q(x)$ as $t \to \infty$ uniformly in $x$ where $g \geqq 0$, $g \not\equiv 0$ and $0 < C^{-1} \leqq q \leqq C < \infty$ for some $C$, then setting $\tilde{V}(x) = V(x, q(x))/q(x)$ we conclude from (4.35) in the same way as in (4.36) that $\lambda_G^\varepsilon(\tilde{V}) = 0$.

## 5. Discrete time models

Suppose that $M$ is a compact metric space, $G \subset M$ is an open set, and $V$ is a continuous function. Consider a Markov chain $X_n$, $n = 0, 1, \ldots$ on $M$ whose transition probabilities

$$P(x, \Gamma) = P\{X_{n+1} \in \Gamma \mid X_n = x\}$$

have continuous positive densities $p(x, y)$ with respect to certain probability measure $m$ on $M$ such that $m(\bar{G}) > 0$. Set $\tau_G = \min\{n : X_n \notin \bar{G}\}$ and introduce the operator

$$\begin{aligned} T_{V,G}\, g(x) &= E_x \chi_{\tau_G > 1}\, g(X_1) \exp(V(X_1)) \\ &= \chi_{\bar{G}}(x) \int_{\bar{G}} p(x, y) e^{V(y)} g(y) dm(y) \end{aligned} \tag{5.1}$$

where $E_x$ is the expectation for the process $X_n$ starting at $x$. The operator $T_{V,G}$ leaves invariant the space $C_0(\bar{G})$ of functions which are continuous in $\bar{G}$ and equal zero outside of $\bar{G}$. Denote by $\|\cdot\|$ the supremum norm for functions and the corresponding norm for operators. If $T_{V,G}(n)$ denotes the $n$-th iterate of $T_{V,G}$, $T_{V,G}(1) = T_{V,G}$, then $T_{V,G}(n) g(x) = E_x \chi_{\tau_G > n}\, g(X_n) \exp(\Sigma_{k=1}^n V(X_k))$ and by the submultiplicative property of the norm the limit

$$\lambda_G(V) = \lim_{n \to \infty} \frac{1}{n} \ln \| T_{V,G}(n) \| \tag{5.2}$$

exists. Since $p(x, y) > 0$ it is easy to see that the limit (5.2) does not depend on the choice of a norm. From [DV2] it follows that

$$\lambda_G(V) = \sup_{\mu \in \mathcal{P}(\bar{G})} \left( \int_{\bar{G}} V d\mu - I_G(\mu) \right) \tag{5.3}$$

where

$$(5.4) \qquad I_G(\mu) = -\inf_{u \in C_+(\bar{G})} \int_G \ln\left(\frac{Pu}{u}\right) d\mu,$$

$Pu(x) = \int_G p(x, y)u(y)dm(y)$, and $C_+(\bar{G})$ is the space of positive continuous functions on $\bar{G}$.

REMARK 5.1. In fact, [DV2] treats the case of a Markov operator and if we restrict $P$ to $\bar{G}$ then we obtain, in general, a sub-Markov operator. From §1, Ch. 7 of [Kr] it follows that there exists a positive eigenfunction $w$ of the operator $P$ corresponding to the eigenvalue $\gamma$ with maximal absolute value. Put $\tilde{p}(x, y) = (w(x)\gamma)^{-1}m(\bar{G})\, p(x, y)w(y)$, $\tilde{m} = (m(\bar{G}))^{-1}m$, $\tilde{P}g(x) = \int \tilde{p}(x, y)g(y)d\tilde{m}(y)$, and $\tilde{T}_{V,G}g(x) = \tilde{P}(e^V g)(x)$. Now we can use [DV2] in order to derive (5.2)–(5.4) for $\tilde{T}_{V,G}$ and $\tilde{P}$ in place of $T_{V,G}$ and $P$. But then we see directly that (5.2)–(5.4) remain true for $T_{V,G}$ and $P$, as well.

From the theory of positive operators (see [Kr], §1 of Ch. 7) it follows that in our circumstances $e^{\lambda_G(V)}$ is a simple eigenvalue of $T_{V,G}$ having maximal absolute value and the corresponding eigenfunction $r_{V,G}$ is positive and continuous on $\bar{G}$.

Introduce a Markov chain $\mathcal{Y}_n$ with the transition operator

$$(5.5) \quad \mathcal{T}_{V,G}\, g(x) = e^{-\lambda_G(V)}(r_{V,G}(x))^{-1} \int_G p(x, y)e^{V(y)} r_{V,G}(y)g(y)dm(y).$$

Since

$$(5.6) \quad I^{\mathcal{T}}(\mu) = -\inf_{u \in C_+(\bar{G})} \int_G \ln\left(\frac{\mathcal{T}_{V,G}u}{u}\right) d\mu = I_G(\mu) - \int V d\mu + \lambda_G(V)$$

then from Lemma 2.5 of [DV2] and the uniqueness of an invariant measure under Delblin's condition we derive

PROPOSITION 5.1. *There exists a unique probability invariant measure $\mu_{V,G}$ of the Markov chain $\mathcal{Y}_n$ which is the only probability measure on $\bar{G}$ satisfying*

$$(5.7) \qquad \lambda_G(V) = \int_{\bar{G}} V d\mu_{V,G} - I_G(\mu_{V,G}).$$

Since the functional $I_G(\mu)$ is convex and lower semicontinuous we can use Proposition 3.2 without any change.

Now consider a family of Markov chains $X_n^\varepsilon$, $\varepsilon > 0$ with transition densities $p^\varepsilon(x, y)$ satisfying the same conditions as $p(x, y)$ above and uniformly in $x \in M$,

$$\int p^\varepsilon(x,y)g(y)dm(y) \to g(Fx) \quad \text{as } \varepsilon \to 0 \tag{5.8}$$

for any $g \in C(M)$ where $F: M \to M$ is a continuous map. Define $T^\varepsilon_{V,G}$ by (5.1) with $X^\varepsilon_1$ in place of $X_1$, define $\lambda^\varepsilon_G$ by (5.2) with $T^\varepsilon_{V,G}$ in place of $T_{V,G}$, and define $I^\varepsilon_G(\mu)$ by (5.4) with $P^\varepsilon u(x) = \int p^\varepsilon(x,y)dm(y)$ in place of $Pu(x)$.

PROPOSITION 5.2. *Suppose that for some sequence of measures $\mu_i \in \mathscr{P}(\bar{G})$ and a sequence of numbers $\varepsilon_i \to 0$ there is a constant $C < \infty$ such that*

$$I^{\varepsilon_i}_G(\mu_i) \leqq C \quad \textit{for all } i = 1, 2, \dots . \tag{5.9}$$

*Then as $i \to \infty$ any weak limit $\mu$ of measure $\mu_i$ will be an invariant measure of the map $F$, i.e. $\mu(F^{-1}U) = \mu(U)$ for any Borel set $U \subset \bar{G}$. So the set $\mathscr{M}^F_G$ of $F$-invariant probability measures with support in $\bar{G}$ is not empty or, equivalently, there exists a $F$-invariant set in $\bar{G}$. If, on the other hand, $\mathscr{M}^F_G$ is empty then*

$$I^{\varepsilon_i}_G(\mu_i) \to \infty \quad \textit{as } i \to \infty \tag{5.10}$$

*for any sequence $\varepsilon_i \to 0$ and $\mu_i \in \mathscr{P}(\bar{G})$. In particular, if $\mu^\varepsilon_{V,G}$ is defined by (uniquely, by Proposition 5.1)*

$$\lambda^\varepsilon_G(V) = \int_G V d\mu^\varepsilon_{V,G} - I^\varepsilon_G(\mu^\varepsilon_{V,G}) \tag{5.11}$$

*and $\mathscr{M}^F_G$ is not empty, then any weak limit as $\varepsilon \to 0$ of measures $\mu^\varepsilon_V$ belong to $\mathscr{M}^F_G$.*

PROOF. If $\mu_i \xrightarrow{w} \mu$ then by the uniform convergence in (5.8),

$$C \geqq \liminf_{i\to\infty} I^{\varepsilon_i}_G(\mu_i) = \liminf_{i\to\infty}\left( - \inf_{u\in C_+(\bar{G})} \int_{\bar{G}} \ln\left(\frac{P^{\varepsilon_i}u}{u}\right) d\mu_i\right)$$

$$\geqq - \inf_{u\in C_+(\bar{G})} \limsup_{i\to\infty} \int_{\bar{G}} \ln\left(\frac{P^{\varepsilon_i}u}{u}\right) d\mu_i \tag{5.12}$$

$$= - \inf_{u\in C_+(\bar{G})} \int_{\bar{G}} \ln\frac{u(Fx)}{u(x)} d\mu(x) \overset{\text{def}}{=} I^F_G(\mu).$$

But

$$I^F_G(\mu) = - \inf_{u\in C_+(\bar{G})} \int_{\bar{G}} \ln\frac{(u(Fx))^n}{(u(x))^n} d\mu(x) = nI^F_G(\mu) \tag{5.13}$$

and since $I^F_G(\mu) \geqq 0$ then either $I^F_G(\mu) = 0$ or $= \infty$ which together with (5.12) gives $I_G^F(\mu)=0$. Applying arguments of Lemma 2.5 of [DV2] in the same way as in Proposition 3.3 we derive that $I_G^F(\mu)=0$ occurs if and only if $\mu$ is $F$-invariant. Other assertions of Proposition 5.2 follow in the same way as in Proposition 3.3. □

THEOREM 5.3. (i) *For any* $V \in C(\bar{G})$

$$\limsup_{\varepsilon\to 0} \lambda_G^\varepsilon(V) \leqq \sup_{\mu\in\mathcal{M}_G^F} \int_{\bar{G}} V d\mu. \tag{5.14}$$

*If* $\mathcal{M}_G^F$ *then one has to put* $-\infty$ *in the right hand side of* (5.14).

*If* $G=M$ *then also*

$$\liminf_{\varepsilon\to 0} \lambda_M^\varepsilon(V) \geqq \inf_{\mu\in\mathcal{M}_G^F} \int_M V d\mu. \tag{5.15}$$

(ii) *Suppose that* $\mathcal{M}_G^F$ *consists of one measure only and either* $G=M$ *or* $K=\operatorname{supp}\mu^0\subset G$, $\bar{G}\neq M$ *and* $K$ *is an attractor for* $F$ *(the definition is the same as in Theorem* 3.6 *with* $F^t$, $t=0,1,2,\dots$ *being iterates of* $F$*). Then for each* $V\in C(\bar{G})$,

$$\lim_{\varepsilon\to 0} \lambda_G^\varepsilon(V) = \int_{\bar{G}} V d\mu^0. \tag{5.16}$$

PROOF. The Proof of (i) is the same as in Theorem 3.5. Hence if $G=M$, (5.16) follows. If $K=\operatorname{supp}\mu^0\subset G$ then the proof of (5.16) can be obtained by a slight modification of proof of Theorem 3.6. By (5.8) for each $\delta>0$,

$$\sup_{x\in M} \int_{M\setminus U_\delta(Fx)} p^\varepsilon(x,y)dm(y) = \gamma(\varepsilon,\delta)\to 0 \quad \text{as } \varepsilon\to 0. \tag{5.17}$$

The main difference with the continuous time case is that we do not have, in general, any estimates of $\gamma(\varepsilon,\delta)$ for small $\varepsilon$ and $\delta$. Still we can choose $n=n(\delta)$ in place of $n(\varepsilon)$ in (3.40) so that with probability close to one paths of the process $X_n^\varepsilon$ turn out to be $\delta^2$-pseudo-orbits of $F$. The remaining part of the proof is similar to Theorem 3.6 since one needs here only the uniform continuity and not smoothness of the map $F$ together with (5.17). □

REMARK 5.2. A version of Theorem 2.1 can be proved already under the above general conditions, but since we not have estimates of $\gamma(\varepsilon,\delta)$ then to go through we have to assume that $t(\varepsilon,\delta)$ in Assumption A satisfies $(t(\varepsilon,\delta))^{-1}\ln\gamma(\varepsilon,r)\to-\infty$ as $\varepsilon\to 0$ for all $\delta, r>0$.

REMARK 5.3. Everything goes through if we require only that $k$-step transition probabilities of $X_n^\varepsilon$ have positive densities for some $k$ which may be bigger than one. Also the compactness assumption on $M$ can be relaxed since we are only interested in the behavior of the processes $X_n^\varepsilon$ in $\bar{G}$. Besides one can consider $X_n^\varepsilon$ whose transition densities $p^\varepsilon(x, y)$ are continuous in $\bar{G}$, positive in $G$, and zero on $\partial G$. In this case one needs an additional assumption saying that $p^\varepsilon(x, y)$ tends to zero as $x \to \partial G$ with the same rate as $\mathrm{dist}(x, \partial G)$ which we had in Section 3 by the strong maximum principle for parabolic equations.

Let now $F$ be a $C^2$-diffeomorphism of a $\nu$-dimensional $C^2$ compact Riemannian manifold $M$. We assume that transition densities $p^\varepsilon(x, y)$ with respect to the Riemannian volume of Markov chains $X_n^\varepsilon$ behave locally as $\varepsilon^{-\nu} q_{F_x}(\varepsilon^{-1} \mathrm{Exp}_{F_x}^{-1} y)$ where $\mathrm{Exp}_z : T_z M \to M$ is the exponential map of the tangent space $T_z M$ to $M$ and $q_z$ is a continuous positive function on $T_z M$ decreasing exponentially fast at infinity and whose integral with respect to the Riemannian volume in $T_z M$ equals one (for more precise conditions on transition densities see Assumptions 1.1 and 7.1 from Ch. 2 of [Ki4]). Recall that a compact $F$-invariant set $K \subset M$ is called hyperbolic if the tangent bundle $TM$ restricted to $K$ can be written as the direct sum of continuous subbundles $T_K M = \Gamma^s \oplus \Gamma^u$ satisfying the exponential dichotomy conditions given before Theorem 2.2. A hyperbolic set $K$ is called basic hyperbolic if the periodic orbits of $F|_K$ are dense in $K$, $F|_K$ is topologically transitive, and there is an open set $U \supset K$ with $K = \bigcap_{-\infty < n < \infty} F^n U$. The topological pressure $P_K(F, \psi)$ of $F$ on $K$ given a continuous function $\psi$ can be characterized by the variational principle

$$P_K(F, \psi) = \sup_{\mu \in \mathcal{M}_K^F} \left( h_\mu(F) + \int_K \psi d\mu \right) \tag{5.18}$$

where $h_\mu(F)$ is the entropy of $F$. Let $J(x)$ be the absolute value of the Jacobian of the linear map $DF : \Gamma_x^u \to \Gamma_{Fx}^u$. Define $\varphi^u(x) = -\ln J(x)$.

THEOREM 5.4. *Let $K$ be a basic hyperbolic set and $U$ be its neighborhood such that $\bar{U}$ contains no other $F$-invariant sets. Then*

$$\lim_{\varepsilon \to 0} \lambda_U^\varepsilon(V) = P_K(F, \varphi^u + V). \tag{5.19}$$

The proof goes through exactly in the same way as in Theorem 2.2 using the machinery of Section 2.7 from [Ki4].

In the same way as Theorem 3.4 we derive from above the following result.

THEOREM 5.5. *Suppose that the maximal $F$-invariant set in $\bar{G}$ is a disjoint union of basic hyperbolic sets $K_i \subset G$, $i = 1, \ldots, \kappa$. Then for each $V \in C(\bar{G})$ all* *limit points as $\varepsilon \to 0$ of measures $\mu_V^\varepsilon$ are equilibrium states $\mu_V^0$ of $F$ corresponding to the function $\varphi^u + V$ (with $\varphi^u$ extended continuously from $K_i$, $i = 1, \ldots, \kappa$ to the whole $\bar{G}$) which are linear combinations of equilibrium states $\mu_{V,K_i}^0$ of $F$ on $K_i$ for those $i$ for which $P_{K_i}(F, \varphi^u + V) = \max_{1 \leqq j \leqq \kappa} P_{K_j}(F, \varphi^u + V)$. If $V$ is Hölder continuous then each $\mu_{V,K_i}^0$ is uniquely defined.*

REMARK 5.4. Though the estimates as in (4.11) of probabilities for the Markov chains $X_n^\varepsilon$ to stay in tube neighborhoods of orbits of $F$ needed to derive (5.19) are more robust than the direct study of absolute continuity of measures $\mu_0^\varepsilon$ in the unstable direction as I did in [Ki4], the method of the present paper needs two-sided bounds of transition densities $p^\varepsilon(x, y)$ as in Assumptions 1.1 and 7.1 of Chapter 2 in [Ki4] and the proof from [Ki4] concerning the limiting behavior of $\mu_0^\varepsilon$ as $\varepsilon \to 0$ uses only upper bounds of $p^\varepsilon(x, y)$ from Assumption 1.1 of [Ki4]. Of course, the method here describes the limiting behavior of $\mu_V^\varepsilon$ for all continuous $V$ which can not be done by the approach of [Ki4].

## 6. Unperturbed operators

In this section we consider the family of unperturbed operators $T_{V,G}^0(t) = T_{V,G}(t)$ for $t \geqq 0$ or $t \leqq 0$ defined by

$$T_{V,G}(t)g(x) = g(F^t x)\chi_{G_t}(x)\exp\left(\int_0^t V(F^s x)ds\right) \tag{6.1}$$

where $G_t = \{x \in G : F^s x \in \bar{G} \text{ for all } s \in [0, t]\}$. Clearly, if $y \in G_{-t}$ then

$$T_{V,G}(-t)(T_{V,G}(t)g)(y) = g(y). \tag{6.2}$$

In the discrete time case the definition is similar,

$$T_{V,G}(n)g(x) = g(F^n x)\chi_{G_n}(x)\exp\left(\sum_{k=1}^{n} V(F^k x)\right), \tag{6.3}$$

where $G_n = \{x \in G : F^k x \in \bar{G} \text{ for all } k \in [0, n]\}$. Both cases can be treated in the same way, so we will use only the representation (6.1) meaning in all statements both the continuous and the discrete time case. As before $G$ may be a proper open subset of $M$ or $G = M$ with $M$ compact and in this case $G_t = M$ for all $t$.

We have seen already that the limit in (1.4) does not depend on the norm because diffusions mix points. On the other hand, we will see that

(6.4)
$$\lambda_G(V)=\lim_{t\to\infty}\frac{1}{t}\ln\|T_{V,G}(t)\|$$

which is the logarithm of the spectral radius of the unperturbed semigroup $T_{V,G}(t)$ depends on the choice of a norm. In what follows we assume that $F^t$ is a continuous flow and that $V$ is a continuous function. For each $x\in M$ let $\delta_x$ denotes a unit mass sitting at $x$. For any family of points $x_t\in G_t$ consider the probability measures

(6.5)
$$\mu_t=\frac{1}{t}\int_0^t\delta_{F^s x_t}\,ds$$

on $\bar{G}$. According to Theorem 6.9 of [W] all weak limits as $t\to\infty$ or as $t\to-\infty$ of measures $\mu_t$ belong to the set $\mathcal{M}_G^F$ of $F^t$-invariant probability measures on $\bar{G}$. For any compact $K\supset\bar{G}$ denote by $\mathcal{N}_{G,K}^{F,+}$ and $\mathcal{N}_{G,K}^{F,-}$ the subsets of $\mathcal{M}_G^F$ of all measures which can be obtained as weak limits of $\mu_t$ for all $x_t\in K\cap G_t$ at $t\to\infty$ or $t\to-\infty$, respectively.

THEOREM 6.1. (a) *If $\mathcal{M}_G^F\neq\varnothing$ and $\lambda_G(V)$ is defined by* (6.4) *with* $\|\cdot\|$ *being the supremum norm then*

(6.6)
$$\lambda_G(V)=\sup_{\mu\in\mathcal{M}_G^F}\int_{\bar{G}}Vd\mu.$$

*If $\mathcal{M}_G^F=\varnothing$ then $\lambda_G(V)=-\infty$. The number $\lambda_G(V)$ remains the same if* $\lim_{t\to\infty}$ *is replaced in* (6.4) *by* $\lim_{t\to-\infty}$.

(b) *Let $\eta\in\mathcal{P}(\bar{G})$ be $F^t$-quasiinvariant in the sense that for any $y\in G_{-t}$ and $t\geqq 0$*

(6.7)
$$\frac{d(\eta\circ F^{-t})(y)}{d\eta(y)}=\rho_t(y),$$

*where $\rho_t(y)$ is a positive continuous function on* supp $\eta$. *Suppose that $\lambda_G(V)=\lambda_G^{(q,\eta)}(V)$ defined by* (6.4) *corresponds to $L^q(\bar{G},\eta)$, $q\geqq 1$-norm $\|\cdot\|_q$ of functions, $\|f\|_q=(\int|f|^q d\eta)^{1/q}$, and the associated operator norm. If $\mathcal{N}_{G,\,\mathrm{supp}\,\eta}^{F,-}\neq\varnothing$ then*

(6.8)
$$\lambda_G^{(q,\eta)}(V)=\sup_{\mu\in\mathcal{N}_{G,\,\mathrm{supp}\,\eta}^{F,-}}\int\left(V+\frac{1}{q}\ln\rho_1\right)d\mu.$$

*If $\mathcal{N}_{G,\,\mathrm{supp}\,\eta}^{F,-}=\varnothing$ then $\lambda_G^{(q,\eta)}(V)=-\infty$. The formula* (6.8) *remains true if one takes there $\mathcal{N}_{G,\,\mathrm{supp}\,\eta}^{F,+}$ in place of $\mathcal{N}_{G,\,\mathrm{supp}\,\eta}^{F,-}$ and replaces $t$ by $-t$ in* (6.4) *and* (6.7).

*If* $\operatorname{supp}\eta \supset \operatorname{supp}\mu$ *for all* $\mu \in \mathcal{M}_G^F$ *(in particular, if* $\operatorname{supp}\eta = \bar{G}$*) then the supremum in* (6.8) *should be taken over the whole* $\mathcal{M}_G^F$. *If* $\eta \in \mathcal{M}_G^F$ *then the supremum in* (6.8) *should be taken over all* $\mu \in \mathcal{M}_G^F$ *satisfying* $\operatorname{supp}\mu \subset \operatorname{supp}\eta$.

PROOF. (a) Since $G_t$, $t \geqq 0$ is a decreasing family of closed sets then if all $G_t$ are non-empty we have $K \overset{\text{def}}{=} \bigcap_{t \geqq 0} G_t \neq \varnothing$. It is easy to see that $F^t K \subset K$ for all $t \geqq 0$ and since $K$ is a compact set there exists $\mu \in \mathcal{M}_G^F$ with $\operatorname{supp}\mu \subset K$. Thus if $\mathcal{M}_G^F = \varnothing$ then for all $t$ large enough $G_t = \varnothing$ and by (6.1) for such $t$, $\| T_{V,G}(t) \| = 0$ which by (6.4) implies that $\lambda_G(V) = -\infty$ in this case.

Now assume that $\mathcal{M}_G^F \neq \varnothing$. By (6.1), if $\| \cdot \|$ is the supremum norm then

$$\| T_{V,G}(t) \| \leqq \sup_{x \in G_t} \exp\left( \int_0^t V(F^s x) ds \right). \tag{6.9}$$

On the other hand, if $g \equiv 1$ in $\bar{G}$ then $\| T_{V,G}(t) g \|$ equals the right hand side of (6.9) and since $\| T_{V,G}(t) \| \geqq \| T_{V,G}(t) g \|$ we see that, in fact, one has in (6.9) the equality. Thus by the compactness of $G_t$ and by the continuity of $V$ there exists $x_t \in G_t$ such that

$$\frac{1}{t} \ln \| T_{V,G}(t) \| = \frac{1}{t} \int_0^t V(F^s x_t) ds = \int V d\mu_t \tag{6.10}$$

where $\mu_t$ is given by (6.5). Since all weak limits of $\mu_t$ as $t \to \infty$ are $F^t$-invariant measures then passing in (6.10) to the limit as $t \to \infty$ we derive

$$\lambda_G(V) \leqq \sup_{\mu \in \mathcal{M}_G^F} \int V d\mu. \tag{6.11}$$

The supremum in the right hand side of (6.11) is attained on some measure $\mu_0 \in \mathcal{M}_G^F$ and we can choose $\mu_0$ to be ergodic since ergodic measures span $\mathcal{M}_G^F$. Then for $\mu_0$-almost all $x$,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t V(F^s x) ds = \int V d\mu_0. \tag{6.12}$$

Each of such $x$ belong to $G_t$ for all $t$. Taking one of these $x$ we obtain

$$\frac{1}{t} \ln \| T_{V,G}(t) \| \geqq \frac{1}{t} \int_0^t V(F^s x) ds$$

and passing here to the limit as $t \to \infty$ we derive an inequality opposite to (6.11), and (6.6) follows. The same proof goes through for negative $t$.

(b) If $\|\cdot\|_q$ is the $L^q(\bar{G},\eta)$-norm with $\eta\in\mathscr{P}(\bar{G})$ satisfying (6.7) then, by (6.1) for any $t\geqq 0$,

$$\|T_{V,G}(t)f\|_q=\left(\int_{G_t}|f(F^tx)|^q\exp\left(q\int_0^t V(F^sx)ds\right)d\eta(x)\right)^{1/q}$$

$$=\left(\int_{G_{-t}}|f(y)|^q\exp\left(q\int_0^t V(F^{-u}y)du\right)\rho_t(y)d\eta(y)\right)^{1/q} \tag{6.13}$$

$$\leqq\|f\|_q\sup_{y\in G_{-t}\cap\operatorname{supp}\nu}(\rho_t(y))^{1/q}\exp\left(\int_0^t V(F^{-u}y)du\right).$$

Since $\rho_t$ is a positive function then it is easy to see from (6.7) and the definition of $G_{-t}$ that $F^{-u}(G_{-t}\cap\operatorname{supp}\eta)\subset G_{-t}\cap\operatorname{supp}\eta$ for any $u\in[0,t]$, and so taking into account that $G_{-t}$ decreases as $t$ increases we conclude that the intersection $\tilde{K}=(\bigcap_{t\geqq 0}G_{-t})\cap\operatorname{supp}\eta$ satisfies $F^{-t}\tilde{K}\subset\tilde{K}$. Thus if $\tilde{K}\neq\varnothing$, which is the case when $G_{-t}\cap\operatorname{supp}\eta\neq\varnothing$ for all $t\geqq 0$, then there exists a $F^t$-invariant measure which can be obtained as a weak limit of measures $\mu_t$ as $t\to-\infty$. Hence, if $\mathscr{N}^{F,-}_{G,\operatorname{supp}\eta}=\varnothing$ then $G_{-t}\cap\operatorname{supp}\eta=\varnothing$ for all $t$ large enough which by (6.13) implies $\|T_{V,G}(t)\|_q=0$, and so $\lambda_G^{(q,\eta)}(V)=-\infty$.

Assume now that $\mathscr{N}^{F,-}_{G,\operatorname{supp}\eta}\neq\varnothing$, and so $G_{-t}\cap\operatorname{supp}\eta\neq\varnothing$ for all $t\geqq 0$. The supremum in the right hand side of (6.13) is attained for some $y=y_0\in G_{-t}\cap\operatorname{supp}\eta$. Choose $f_\varepsilon$ in (6.13) so that $f_\varepsilon(y)=(\eta(U_\varepsilon(y_0)))^{1/q}$ when $y\in U_\varepsilon(y_0)=\{z:\operatorname{dist}(z,y_0)<\varepsilon\}$ and $f_\varepsilon\equiv 0$ outside of $U_\varepsilon(y_0)$. Since $y_0\in\operatorname{supp}\eta$ then $\eta(U_\varepsilon(y_0))>0$, and so $\|f_\varepsilon\|_q=1$. For small $\varepsilon$ we can get $\|T_{V,G}(t)f_\varepsilon\|_q$ arbitrarily close to the supremum in the right hand side of (6.13) yielding

$$\|T_{V,G}(t)\|_q=\sup_{y\in G_{-t}\cap\operatorname{supp}\eta}(\rho_t(y))^{1/q}\exp\left(\int_0^t V(F^{-u}y)du\right). \tag{6.14}$$

Remark that for any $y\in G_{-(t+s)}$,

$$\rho_{t+s}(y)=\rho_s(F^{-t}y)\rho_t(y), \tag{6.15}$$

i.e. $\rho_t$ is a multiplicative cocycle, and so for any $y\in G_{-t}$ and an integer $m>0$,

$$\rho_t(y)=\rho_{t-n2^{-m}}(F^{-n2^{-m}}y)\rho_{2^{-m}}(F^{-(n-1)2^{-m}}y)\cdots\rho_{2^{-m}}(F^{-2^{-m}}y)\rho_{2^{-m}}(y) \tag{6.16}$$

where $n$ is the integral part of $2^m t$. For $t=n2^{-m}$ the supremum in the right hand side of (6.14) is attained for some $y_{n,m}\in G_{-n2^{-m}}\cap\operatorname{supp}\eta$. Thus by (6.4), (6.14) and (6.16),

$$\lambda_G^{(q,\eta)}(V) = \limsup_{n\to\infty} \frac{1}{n2^{-m}}\left(\frac{1}{q}\ln \rho_{n2^{-m}}(y_{n,m}) + \int_0^{n2^{-m}} V(F^u y_{n,m})du\right)$$

$$\times \limsup_{n\to\infty} \frac{1}{n2^{-m}}\left(\sum_{k=0}^{n-1}\left(\frac{1}{q}\ln \rho_{2^{-m}}(F^{-k2^{-m}}y_{n,m}) + V_m(F^{-k2^{-m}}y_{n,m})\right)\right) \tag{6.17}$$

where

$$V_m(z) = \int_0^{2^{-m}} V(F^{-u}z)du. \tag{6.18}$$

It follows from (6.17) and Theorem 6.9 of [W] that

$$\begin{aligned}\lambda_G^{(q,\eta)}(V) &\leqq \sup_{\mu\in\tilde{\mathcal{N}}_m}\left(\frac{2^m}{q}\int \ln \rho_{2^{-m}}d\mu + 2^m\int V_m d\mu\right)\\ &= \sup_{\mu\in\tilde{\mathcal{N}}_m}\left(\frac{1}{q}\int \ln \rho_1 d\mu + \int V d\mu\right)\end{aligned} \tag{6.19}$$

where $\tilde{\mathcal{N}}_m$ is the set of $F^{2^{-m}}$-invariant measures which can be obtained as weak limits when $n\to\infty$ of measures

$$\mu^{(n,m)} = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{F^{-k2^{-m}}y_n} = \frac{1}{n2^{-m}}\sum_{k=0}^{n-1}2^{-m}\delta_{F^{-k2^{-m}}y_n} \tag{6.20}$$

for all $y_n \in G_{-n2^{-m}} \cap \operatorname{supp}\eta$. On the other hand if $\mu\in\tilde{\mathcal{N}}_m$ and $\mu^{(n_i,m)} \xrightarrow{w} \mu$ as $n_i\to\infty$ then, by (6.4) and (6.14),

$$\begin{aligned}\lambda_G^{(q,\eta)}(V) &\geqq \limsup_{n_i\to\infty} 2^m\int\left(\frac{1}{q}\ln\rho_{2^{-m}} + V_m\right)d\mu^{(n_i,m)}\\ &= 2^m\int\left(\frac{1}{q}\ln\rho_{2^{-m}} + V_m\right)d\mu = \int\left(\frac{1}{q}\ln\rho_1 + V\right)d\mu.\end{aligned} \tag{6.21}$$

Since (6.21) holds true for any $\mu\in\tilde{\mathcal{N}}_m$ it gives, together with (6.19),

$$\lambda_G^{(q,\eta)}(V) = \sup_{\mu\in\tilde{\mathcal{N}}_m}\int\left(\frac{1}{q}\ln\rho_1 + V\right)d\mu. \tag{6.22}$$

We remark that for any continuous function $q$ the integrals $\int g d\mu^{(n,m)}$ are close uniformly in $n$ to $(1/t)\int_0^t g(F^u y_n)du$ for large $m$ and $t = n2^{-m}$. From this we conclude that any weak limit of a sequence $\mu_i \in \tilde{\mathcal{N}}_{m_i}$ with $m_i\to\infty$ belongs to $\mathcal{N}^{F,-}_{G,\operatorname{supp}\eta}$. This together with (6.19) yields (6.8).

Finally, if $\mathrm{supp}\,\eta \supset \mathrm{supp}\,\mu$ for all $\mu \in \mathcal{M}_G^F$ then $\mathcal{N}_{G,\mathrm{supp}\,\eta}^{F,-}$ contains all ergodic measures from $\mathcal{M}_G^F$ and since they span $\mathcal{M}_G^F$ we can take the supremum in (6.8) over the whole $\mathcal{M}_G^F$. If $\eta \in \mathcal{M}_G^F$ then $\mathcal{N}_{G,\mathrm{supp}\,\eta}^{F,-}$ contains all ergodic measures $\mu \in \mathcal{M}_G^F$ with $\mathrm{supp}\,\mu \subset \mathrm{supp}\,\eta$. Since these measures span the space $\mathcal{M}_{\mathrm{supp}\,\eta}^F = \{\mu \in \mathcal{M}_G^F : \mathrm{supp}\,\mu \subset \mathrm{supp}\,\eta\}$ we can take the supremum in (6.8) over $\mathcal{M}_{\mathrm{supp}\,\eta}^F$.

Theorem 6.1 shows that the spectral radius of the semigroup $T_{V,G}(t)$ is not connected with the topological pressure. In the same way as in item (a) of Theorem 6.1 one derives that

$$\lim_{t\to\infty} \frac{1}{t} \ln \| T_{V,G}(t)1 \| = \sup_{\mu \in \mathcal{M}_G^F} \int V d\mu \tag{6.23}$$

where $\|\cdot\|$ is the supremum norm and we apply $T_{V,G}(t)$ to the function $1(x) \equiv 1$. On the other hand, the following result shows that, taking in (6.23) integral norms, we can end up with the topological pressure. For $\eta \in \mathcal{P}(\bar{G})$, $t \geqq 0$ and $x \in G_t$ put $U_\delta^{(\eta)}(x, t, G) = \{y \in G_t \cap \mathrm{supp}\,\eta : \mathrm{dist}(F^s x, F^s y) \leqq \delta$ for all $s \in [0, t]\}$.

THEOREM 6.2. *Suppose that $\eta \in \mathcal{P}(\bar{G})$ and for some continuous function $\varphi$ on* $\mathrm{supp}\,\eta$ *and all $t, \delta > 0$, $x \in G_t \cap$* $\mathrm{supp}\,\eta$ *one has*

$$(A_\delta(t))^{-1} \leqq \eta(U_\delta^{(\eta)}(x, t, G)) \exp\left(-\int_0^t \varphi(F^s x) ds\right) \leqq A_\delta(t) \tag{6.24}$$

*where $A_\delta(t) > 0$ satisfies*

$$\lim_{t\to\infty} \frac{1}{t} \ln A_\delta(t) = 0. \tag{6.25}$$

*If $\eta \in \mathcal{M}_G^F$ then*

$$\lim_{t\to\infty} \frac{1}{t} \ln \| T_{V,G}(t)1 \|_q = \frac{1}{q} P_{\mathrm{supp}\,\eta}(F, \varphi + qV) \tag{6.26}$$

*where $P_K(F, \psi)$ is the topological pressure defined by* (1.19). *If $\eta$ is not necessarily $F^t$-invariant but*

$$\mathrm{supp}\,\eta \supset \bigcup_{\mu \in \mathcal{M}_G^F} \mathrm{supp}\,\mu \stackrel{\mathrm{def}}{=} K, \tag{6.27}$$

*then*

$$\lim_{t\to\infty} \frac{1}{t} \ln \| T_{V,G}(t)1 \|_q = \frac{1}{q} P_K(F, \varphi + qV). \tag{6.28}$$

PROOF. For each $\delta, t > 0$ denote by $S(\delta, t)$ a maximal $(\delta, t)$-separated set (see Section 4) of $F^s$ in $G_t \cap \operatorname{supp} \eta$. Then

$$\bigcup_{x\in S(\delta,t)} U_\delta^{(\eta)}(x, t, G) = G_t \cap \operatorname{supp} \eta \supset \bigcup_{x\in S(\delta,t)} U_{\delta/2}^{(\eta)}(x, t, G) \tag{6.29}$$

and for any $y, z \in S(\delta, t)$, $y \neq z$ the sets $U_{\delta/2}^{(\eta)}(y, t, G)$ and $U_{\delta/2}^{(\eta)}(z, t, G)$ are disjoint. Then

$$\begin{aligned} &\left(\sum_{x\in S(\delta,t)} \eta(U_\delta^{(\eta)}(x, t, G))\exp\left(q\int_0^t (V(F^s x) + \gamma_G(V, \delta))ds\right)\right)^{1/q} \\ &\geqq \| T_{V,G}(t)1 \|_q \\ &\geqq \left(\sum_{x\in S(\delta,t)} \eta(U_{\delta/2}^{(\eta)}(x, t, G))\exp\left(q\int_0^t (V(F^s x) - \gamma_G(V, \delta))ds\right)\right)^{1/q} \end{aligned} \tag{6.30}$$

with $\gamma_G(\psi, \delta)$ defined by (4.10). This together with (6.24) give

$$\begin{aligned} &A_\delta^{1/q}(t)e^{t\gamma_G(V,\delta)} \| T_{V,G}(t)1 \|_q \left(\sum_{x\in S(\delta,t)} \exp\left(\int_0^t (\varphi(F^s x) + qV(F^s x))ds\right)\right)^{-1/q} \\ &\geqq A_{\delta/2}^{-1/q}(t)e^{-t\gamma_G(V,\delta)}. \end{aligned} \tag{6.31}$$

If $\eta \in \mathcal{M}_G^F$ then, taking ln in (6.31), dividing by $t$, letting first $t \to \infty$ and then $\delta \to 0$, we derive (6.26) in view of (6.25) and (4.16)–(4.18). Here we need a standard version of the variational principle (Theorem 9.10 in [W]) for the restriction of the flow $F^t$ to the $F^t$-invariant set supp $\eta$. If $\eta$ may be not $F^t$-invariant but (6.27) holds true, then we derive (6.28) from (6.31) by a slight modification of the proof of Theorem 9.10 in [W]. $\square$

COROLLARY 6.1. *Suppose that the maximal invariant set in $\bar{G}$ of a $C^2$ flow $F^t$ consists of a unique basic hyperbolic set $K \subset G$. If $\eta$ is an equilibrium state corresponding to a Hölder continuous function $\psi$ on $K$ then* (6.26) *holds true with $\varphi = \psi - P_K(F, \psi)$. If $\eta$ is the normalized Riemannian volume on some neighborhood of $K$ then* (6.28) *holds true with $\varphi = \varphi^u$ defined by* (2.5).

PROOF. If $\eta$ is an equilibrium state for a Hölder continuous $\psi$, then it follows from Section 3 of [BR] that the conditions (6.24) and (6.25) will be satisfied for $\varphi = \psi - P_K(F, \psi)$. If $\eta$ is the normalized Riemannian volume on a neighborhood of $K$, then by the volume lemma (Lemma 4.2 of [BR]) the conditions (6.24) and (6.25) will be satisfied for $\varphi = \varphi^u$. □

REMARK 6.1. Corollary 6.1 generalizes Theorem 1 from [V].

REMARK 6.2. Using volume estimates from [Y] one can generalize Corollary 6.1 for uniformly partially hyperbolic sets.

REMARK 6.3. It would be interesting to generalize Corollary 6.1 to the case when $G$ contains several basic hyperbolic sets in the spirit of Theorems 2.1 and 3.4. Namely, let $G = M$ be compact and suppose that $F^t$ is an Axiom A flow with no cycle property, i.e. the maximal $F^t$-invariant set consists of finite number of basic hyperbolic sets $K_1, \ldots, K_\kappa$ which are different equivalence classes in the sense of Section 2. If $\lambda_G^{(q)}(V)$ corresponds to $L^q(M, m)$-norm by the formula (6.4) within the Riemannian volume, then it is quite plausible that

$$\lambda_G^{(q)}(V) = \frac{1}{q} P_M(F, \varphi^u + qV) = \frac{1}{q} \max_{1 \leq i \leq \kappa} P_{K_i}(F, \varphi^u + qV).^\dagger$$

This formula is certainly true and not difficult to prove if all $K_i$, $i = 1, \ldots, \kappa$ are fixed points or periodic orbits of $F^t$. This can be done by a method similar to [Wi].

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